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Elementary Cellular Automata with Elementary Memory Rules in Cells: The Case of Linear Rules

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Standard Cellular Automata (CA) rules depend on the neighborhood configuration only at the preceding time step. This article considers an extension to the standard framework of CA which implements memory capabilities by featuring each cell by elementary rules of its last three states. A study is made of the effect of memory rules on the elementary linear (or additive) CA rules 90 and 150.

1 INTRODUCTION

Cellular Automata (CA) are discrete, spatially explicit extended dynamic systems. A CA system is composed of adjacent cells or sites arranged as a regular d-dimensional lattice, which evolves in discrete time steps. Each cell is characterized by an internal state whose value belongs to a finite set. The updating of these states is made simultaneously according to a common local transition rule involving only a neighborhood of each cell. Thus, if $\sigma_i^{(T)}$ is taken to denote the value of cell i at time step T, the site values evolve by iteration of the mapping: $\sigma_i^{(T+1)} = \phi\ \sigma_j^{(T)} \in \mathcal{N}_i$, where ϕ is an arbitrary function which specifies the cellular automaton *rule* operating on the neighborhood ($\mathcal{N}$) of the cell i ([1, 2]).

Historic memory can be embedded in the CA dynamics by featuring every cell by a mapping of its states in the previous time steps. Thus, what is here proposed is to maintain the transition rules (ϕ) unaltered, but make them act on the cells featured by a function of their previous states:

$\sigma_i^{(T+1)} = \phi\left(s_j^{(T)} \in \mathcal{N}_i\right)$, $s_i^{(T)}$ being a state function of the series of states of the cell i after time-step T.

In an earlier work ([4]-[13]) we analyzed the effect of memory when cells are featured by a weighted mean value (m) of *all* their previous states: $s_i^{(T)} = m\left(\sigma_i^{(1)}, \sigma_i^{(2)}, \ldots, \sigma_i^{(T)}\right)$. Recently [3] we analyzed the effect of a particular type of limited trailing memory in which cells are featured by the most frequent state of their three last states: $s_i^{(T)} = mode\left(\sigma_i^{(T-2)}, \sigma_i^{(T-1)}, \sigma_i^{(T)}\right)$. Here we generalize the study by considering a general mapping (f) of the last three values: $s_i^{(T)} = f\left(\sigma_i^{(T-2)}, \sigma_i^{(T-1)}, \sigma_i^{(T)}\right)$. In the approach adopted here, memory becomes operative after $T = 3$, with the initial assignations $s_i^{(1)} = \sigma_i^{(1)}$, $s_i^{(2)} = \sigma_i^{(2)}$. Cellular automata implementing memory in cells will be termed *historic*, and the standard ones *ahistoric*. As stated, the *historic* and *ahistoric* evolving patterns are the same up to $T = 3$.

We consider here only one-dimensional ($d = 1$) rules in the simplest scenario, thus *elementary* rules: CA with two possible values ($k = 2$) at each site $\left(\sigma \in \{0, 1\}\right)$, with rules operating on nearest neighbors ($r = 1$). Following Wolfram's notation, these rules are characterized by a sequence of binary values (β) associated with each of the eight possible triplets $\left(\sigma_{i-1}^{(T)}, \sigma_i^{(T)}, \sigma_{i+1}^{(T)}\right)$:

111 110 101 100 011 010 001 000

$$(\beta_1 \;\; \beta_2 \;\; \beta_3 \;\; \beta_4 \;\; \beta_5 \;\; \beta_6 \;\; \beta_7 \;\; \beta_8)_{binary} \equiv \left(\sum_{i=1}^{8} \beta_i 2^{8-i}\right)_{decimal} = R \in [0, 255]$$

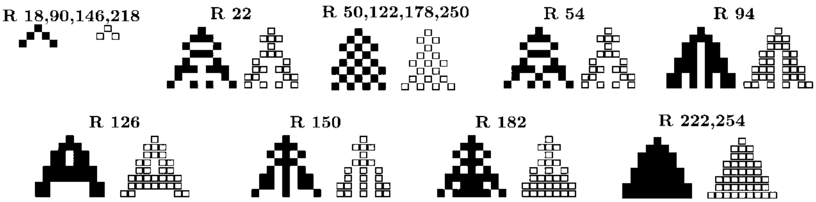
The rules are conveniently specified by a decimal integer, to be referred to as their *rule number, R*.

We pay particular attention to *legal* rules: rules that are *reflection symmetric* ($\beta_2 = \beta_5$ and $\beta_4 = \beta_7$) and *quiescent* ($\beta_8 = 0$). These restrictions leave 32 possible *legal* rules of binary form: $\beta_1\beta_2\beta_3\beta_4\beta_2\beta_6\beta_4 0$.

2 A SIMPLE EXAMPLE: MAJORITY MEMORY

A simple way of tracing the effect of memory is that of featuring cells by their most frequent state, i.e., by means of the *majority* rule f=232 (11101000). Table 1 shows the σ and f spatio-temporal patterns of the legal rules affected by this majority memory mechanism when starting from a single site live cell. Rule ϕ = 254 (1111110) is an easy rule to track: it assigns a live state to any cell in whose neighborhood there would be at least one live cell. In the ahistoric context, rule 254 progresses as fast as possible from a single site live cell, generating segments whose size increases two units with every time step. The dynamics is slower in the

TABLE 1
Legal rules affected by the majority memory rule (R 232) when starting from a single site live cell. Evolution up to $T = 8$. Live cells: last ($\sigma = 1$), most frequent ($s = 1$).

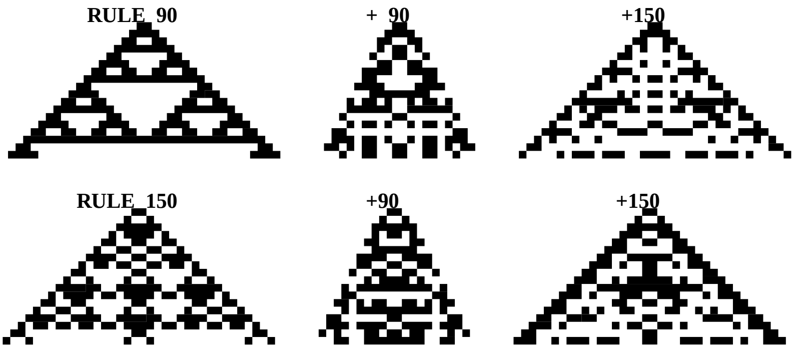


historic majority model (see Table 1): the outer live cells are not featured as live unless they *lived* twice in the last three time-steps, in which case the automaton fires two new outer live cells. Thus the segment of live cells grows only under ϕ=254, f=232 at odd time steps and remains the same at even time steps: the *speed of light* in the historic model is half that of the ahistoric. The outer live cells of Rules 94 and 126 evolve as with Rule 254. The effect of memory on Rules 18,90, 146 and 218 is dramatic: after $T = 3$ the most frequent state of all the cells is the dead one, so these rules lead to extinction at $T = 4$. The *inhibitory* effect of the *majority* memory rule is reported in [3].

It should be emphasized that the memory mechanism considered here is different from that of other CA with memory reported in the literature. Typically, higher-order-in-time rules incorporate memory into the transition rule, determining the configuration at time $T + 1$ in terms of the configurations at previous time-steps. Thus, in second order in time (memory of capacity two) rules, the transition rule operates as: $\sigma_i^{(T+1)} = \Phi\ \sigma_j^{(T)} \in \mathcal{N}_i, \sigma_j^{(T-1)} \in \mathcal{N}_i$. *Double* memory (in transition rule and in cells) can be implemented as: $\sigma_i^{(T+1)} = \Phi\ s_j^{(T)} \in \mathcal{N}_i, s_j^{(T-1)} \in \mathcal{N}_i$. Particularly interesting is the reversible formulation based on the exclusive OR (XOR, noted $\oplus$):$\sigma_i^{(T+1)} = \phi\ \sigma_j^{(T)} \in \mathcal{N}_i\ \oplus \sigma_i^{(T-1)}$, reversed as $\sigma_i^{(T-1)} = \phi\ \sigma_j^{(T)} \in \mathcal{N}_i\ \oplus \sigma_i^{(T+1)}$. Reversion turns out unpracticable when adding memory in all cells to this ahistoric reversible formulation as: $\sigma_i^{(T+1)} = \phi\ s_j^{(T)} \in \mathcal{N}_i\ \oplus s_i^{(T-1)}$. To preserve reversibility, inherent in the XOR operation, the reversible formulation with memory must be: $\sigma_i^{(T+1)} = \phi\ s_j^{(T)} \in \mathcal{N}_i\ \oplus \sigma_i^{(T-1)}$ [6].

Some authors, for example Wolf-Gladrow [17], define rules with *memory* as those with dependence in ϕ on the state of the cell to be updated. So in the $r = 1$ scenario, rules with no *memory* take the form: $\sigma_i^{(T+1)} = \phi\ \sigma_{i-1}^{(T)}, \sigma_{i+1}^{(T)}$. Rule 90 : $\sigma_i^{(T+1)} = \sigma_{i-1}^{(T)} + \sigma_{i+1}^{(T)}$ *mod* 2, would be an example of a rule with no *memory*. Our use of the term memory is not this.

TABLE 2
Rules 90 and 150 remain additive in the linear memory model. The evolution patterns starting with two adjacent live cells as shown in this table coincide with the XOR superposed configuration of those evolved independently starting with a single seed, shown in Figs. 1 and 2. Evolution up to $T = 18$.



CA with memory in cells are cited by Wuensche and Lesser ([15],p.15), who refuse to enter into its study and state that "CA with memory in cells would result in a qualitatively different behavior".

3 THE EFFECT OF ELEMENTARY MEMORY RULES ON ELEMENTARY LINEAR RULES

In the standard (ahistoric) scenario, Rules 90 and 150 (together with the trivial 0 and 204) are the only *linear* (or *additive*) legal rules: i.e., any initial pattern can be decomposed into the superposition of patterns from a single site seed. Each of these configurations can be evolved independently and the results superposed (module two) to obtain the final complete pattern. As illustrated in Table 2, starting with two adjacent live cells, the additivity of Rules 90 and 150 remains valid in the historic model with linear memory rules.

Figures 1 and 2 show the effect of memory on rules 90 and 150 starting from a single site live cell up to $T = 13$[1]. Complementary memory rules[2] have the same effect on rule 90 (regardless of the role played by the three last states in f and the initial configuration). This is why Fig. 1 shows only the effect of memory rules up to f=127. A subset of rules does not affect rule 90; grouped as pair of complementary rules they are: (19,236), (27,228), (51,204), (59,196), (68,187), (76,179), (100,155) and (108,147). Rule 150 is affected by every memory rule except, of course, by the *identity* rule 204(11001100), which assigns $s_i = \sigma_i$ and therefore does

[1] With $\sigma_i^{(T-2)}$ acting as $\sigma_{i-1}^{(T)}$ and $\sigma_i^{(T-1)}$ as $\sigma_{i+1}^{(T)}$ as arguments of f.
[2] Rules whose rule number adds 255.

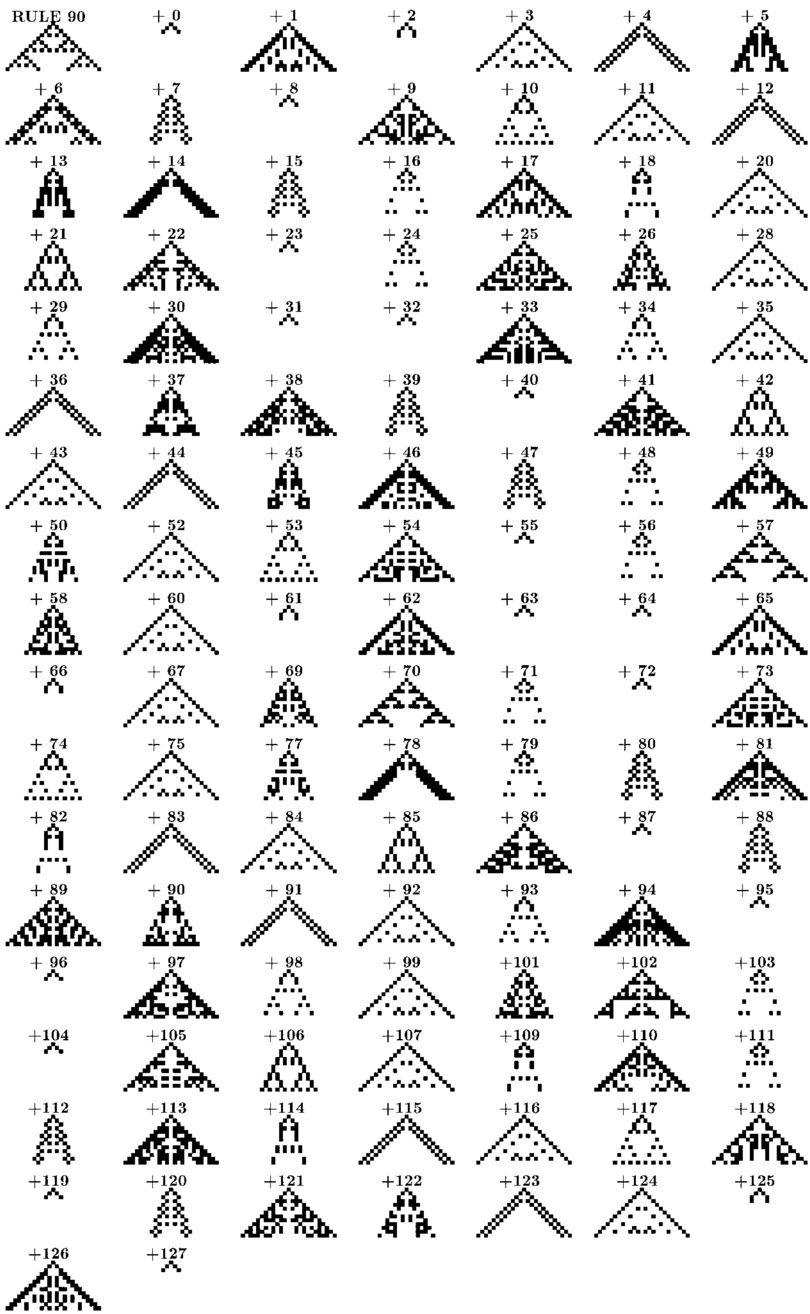


FIGURE 1
Effect of memory rules on rule 90 when starting from a single site seed up to $T = 13$.

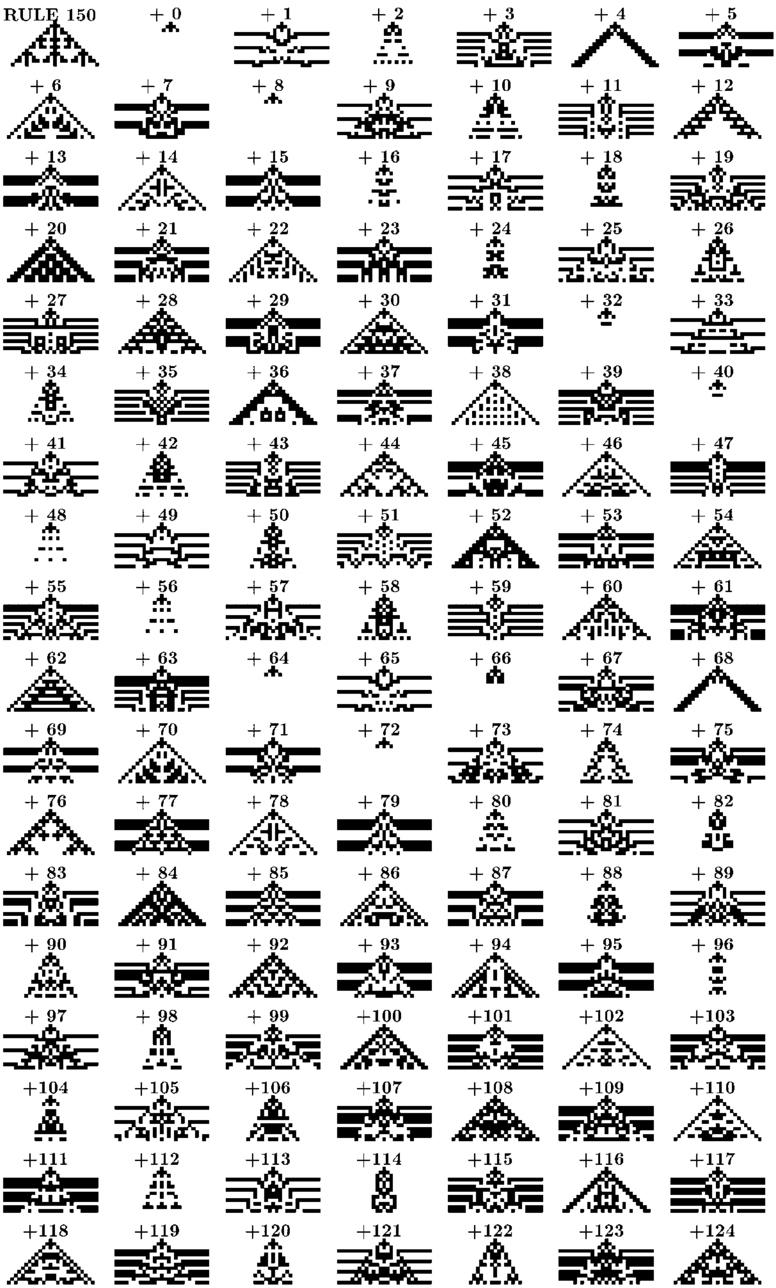


FIGURE 2
Effect of memory rules on rule 150 when starting from a single site seed up to $T = 13$.

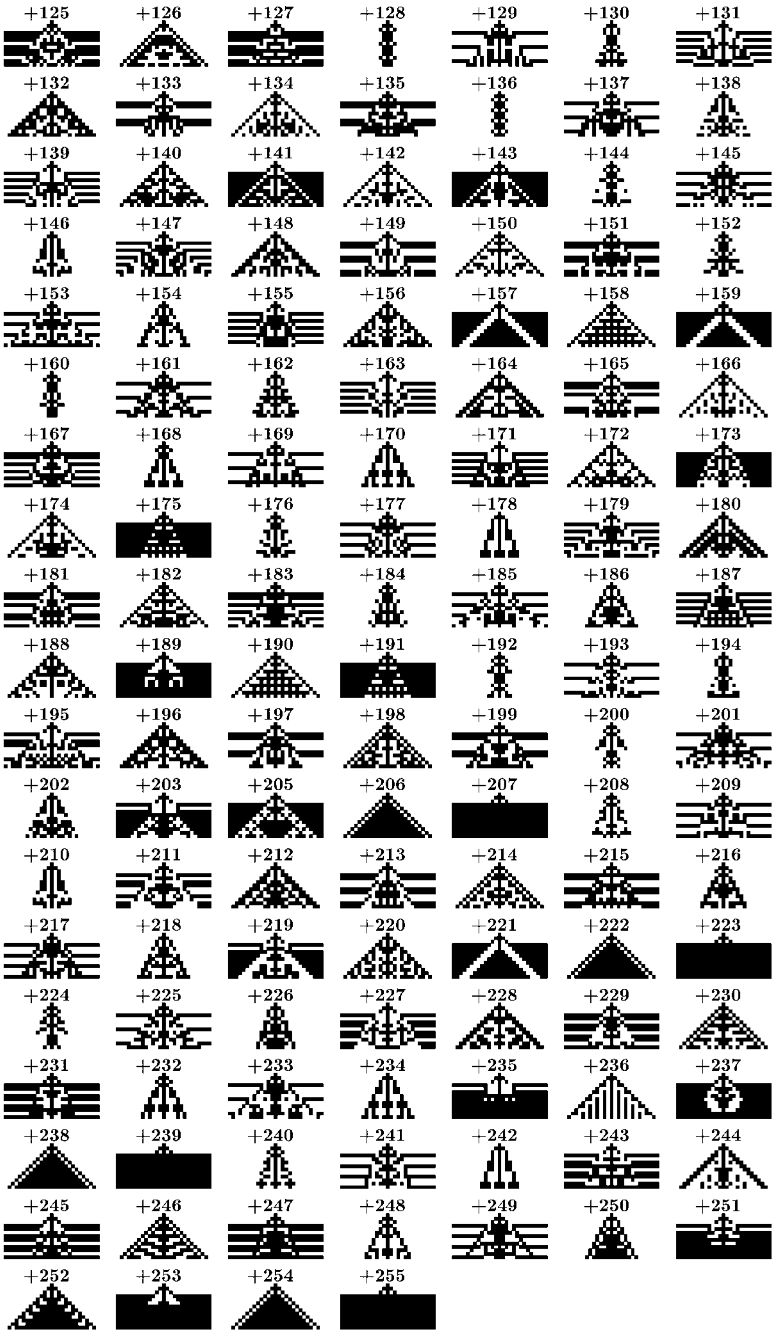


FIGURE 2
Continued.

not affect any rule. Most of the rules that extinguish starting from a single site seed, lead to extinction (e.g., the legal rules 32, 72, 104, 128, 160, 200 and 232 as early as at $T = 4$) or produce patterns consisting of only two *branches* (e.g., +4) when acting as memory rules on rule 90. Rule 150 is much more resilient to extinction (only five rules lead its extinction), or to the formation of two branches (+4, +68). Some patterns with memory are reminiscent of the ahistoric ones, e.g., (90,+20) or (150,+100), but as a rule, memory notably alters the spatio-temporal patterns. As a rule, no general relevant concordance can be traced between the effect of *all* rules in the same equivalence class (formed under the negative, reflection and negative + reflection transformations [15]). Anyway, i) reflected rules tend to produce similar memory effect, e.g., $\{8,64\}$, $\{13,69\}$, $\{29,71\}$, $\{30,86\}$, or $\{184,226\}$, ii) rules in some equivalence classes produce a similar effect when acting as memory rules, e.g., the important rule 110 and its three equivalent rules 124,137 and 193, or those rules in the class {60, 102, 153, 195}. The counterexample is seen in rules $\{19,55\}$ regarding their effect on rule 90: rule 19 does not alter rule 90, rule 55 leads to its extinction at $T = 4$.

Figures 3 and 4 show the effect of legal memory rules on rules 90 and 150, starting from the same random initial configuration[3] up to $T = 150$. These figures also show superimposed the differences produced in patterns (DP) when reversing the value of its initial center site. The illustrations show the *damaged* region as darker pixels corresponding to the site values that differed among the patterns generated with the two initial configurations. Some features of the patterns in Figs. 3 and 4 could be predicted from Figs. 1 and 2. For example, rules 4, 32, 36, 72, 128, 132, 160, 164, and 200 when acting as memory rules on rule 90 lead either to extinction, to the formation of periodic patterns, or they produce soliton-like structures. Nevertheless, the prediction fails for rules such as 104 and 232, or 76, 108 and 236 which do not alter rule 90 starting from a single site seed but notably alter its pattern when starting at random. The effect of rules 4, 32, and 72 on rule 150 in Fig.4 agrees qualitatively with that on Fig.4. The constrained patterns of rules 128, 160, and 200 seem to advance their particular patterns in Fig.4. Rule 36, which exemplifies the type of simple rules which serve as *filters* (Wolfram's Class II), turns out to be sophisticated when acting on rule 150. More predictable is the "complex" effect induced by Rule 54[4] on both rules.

[3]The value of each site is initially uncorrelated, and is taken to be 0 or 1 with probability $p = 0.5$.

[4]The behavior of Rule 54 in the ahistoric model has been featured to some extent as transitional between very simple Wolfram's class I and II rules and chaotic Class III. Thus, Rule 54 appears among the two one-dimensional rules (with Rule 110) that seem to belong to Wolfram's *complex* Class IV.

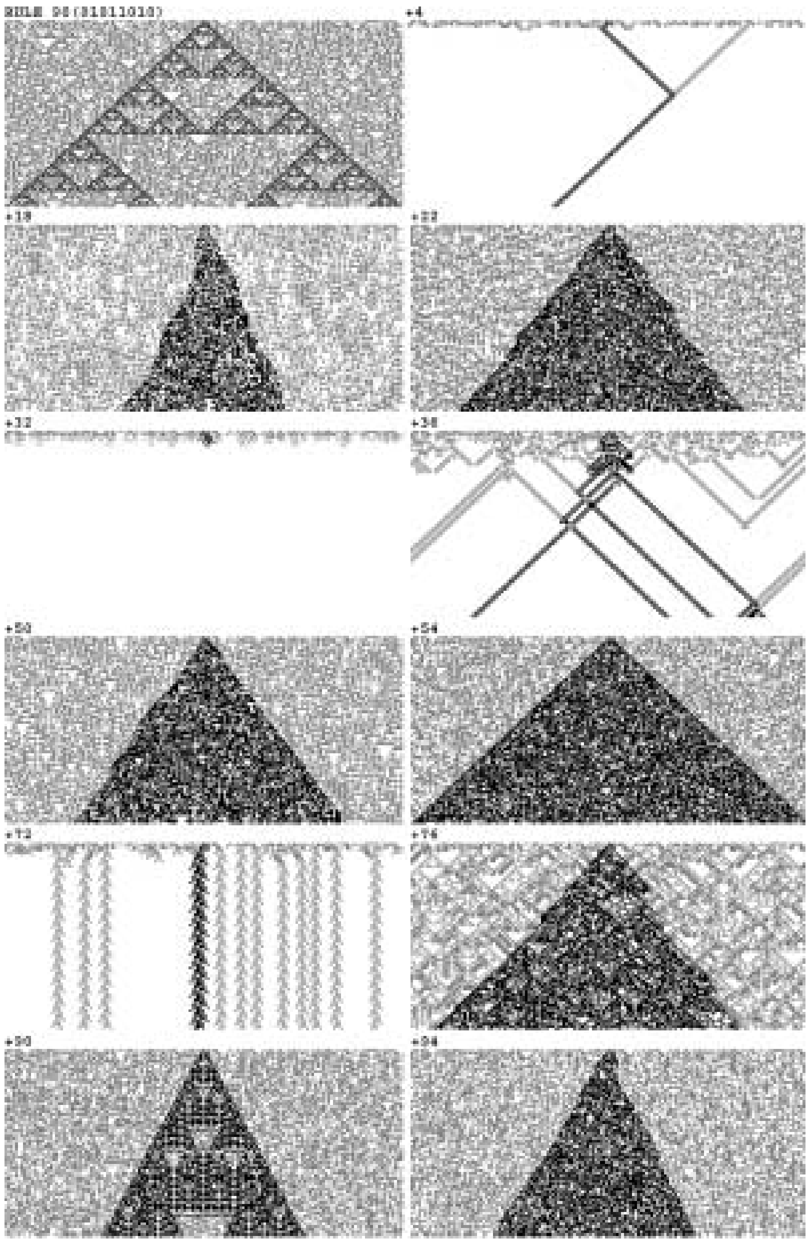


FIGURE 3
Effect of elementary legal rules on rule 90 when starting at random. The values of sites in the initial configuration are chosen to be 0 or 1 with probability 0.5. The simulation was conducted in a lattice of size 211 up to $T = 105$ with periodic boundary conditions imposed on the edges. Differences in patterns resulting from reversing the center site value are shown as darker pixels.

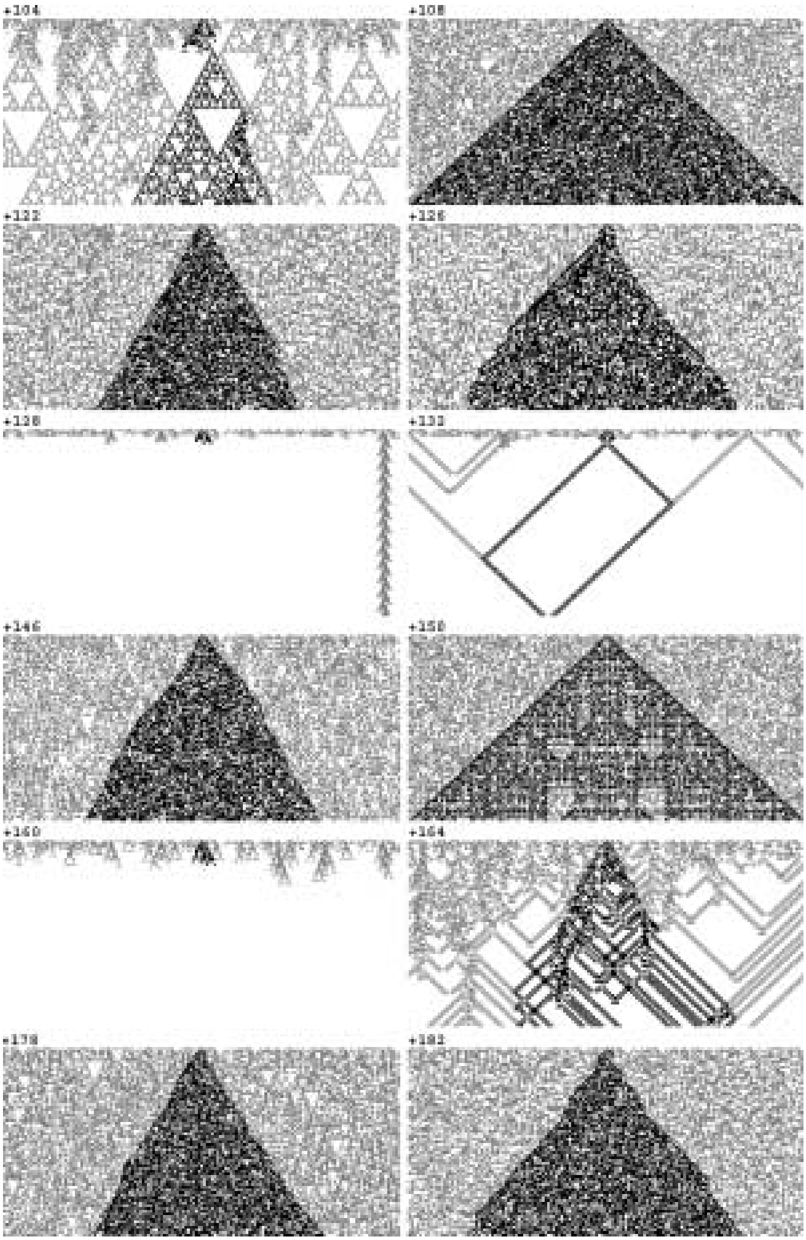


FIGURE 3
Countinued.

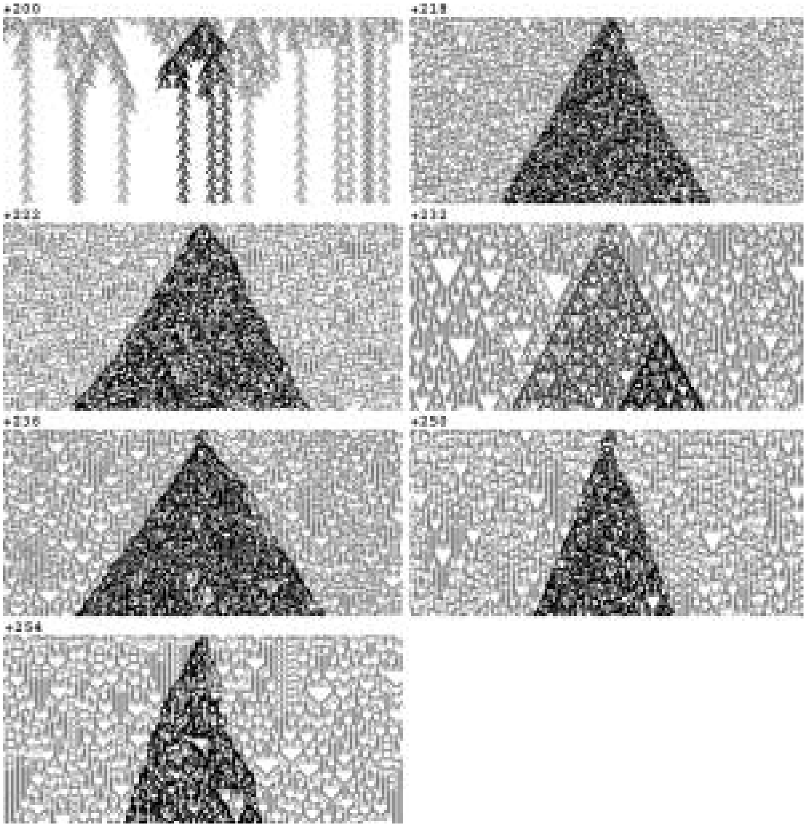


FIGURE 3
Countinued.

4 CONCLUSION

The effect on the linear rules 90 and 150 of elementary memory rules embedded in cells is qualitatively (pictorically) studied in this work. As a rule, memory notably alters the dynamics of ahistoric rules. A more complete analysis of the effect of memory on rules 90 and 150 is left for future work. For example, their dynamics in the state (phase) space or the potential fractal features are to come under scrutiny ([19]). Another interesting open question is the analysis of the effect of memory on solving computational tasks such as synchronization, density and ordering. Anyhow, we have explored the behavior of 255×2 of the 255^2 elementary combinations available. Moreover, the whole CA paradigm can be endowed with memory in cells, e.g., two-dimensional and probabilistic CA. So much remains to be done.

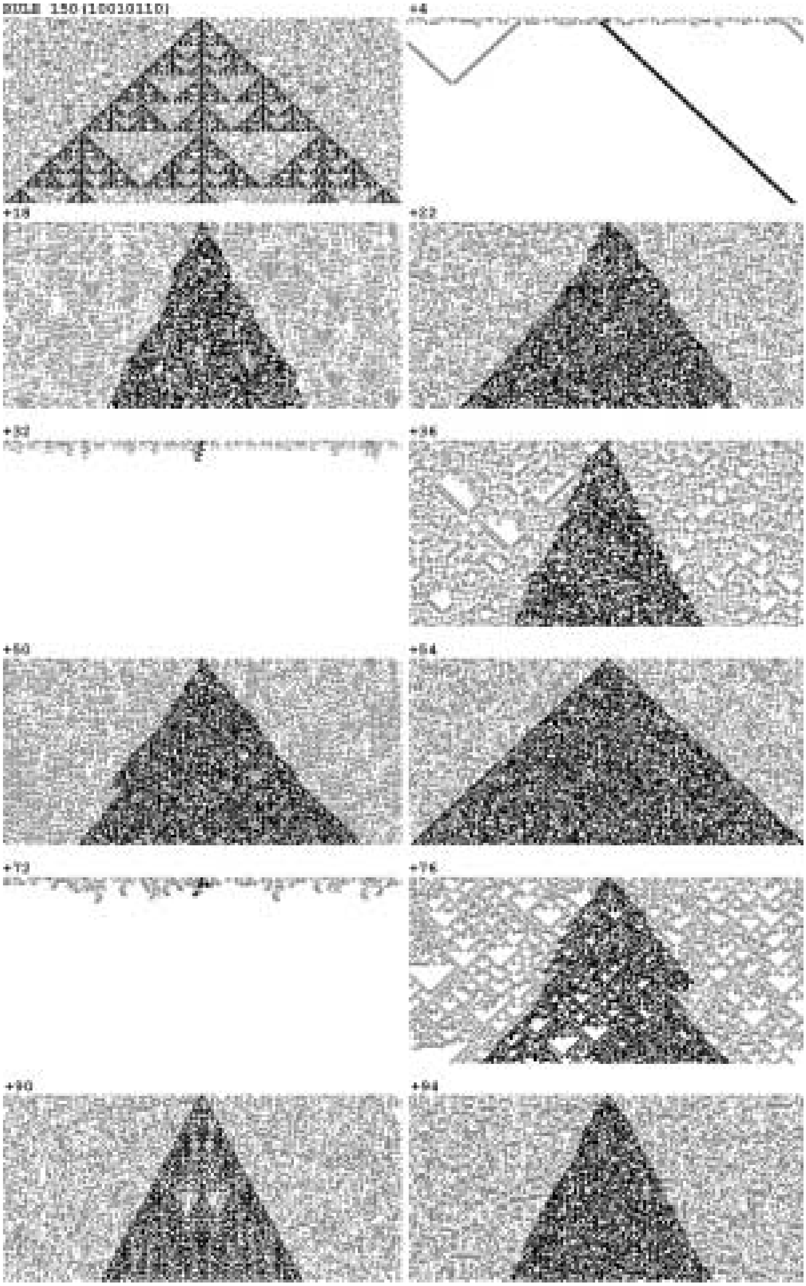


FIGURE 4
Effect of elementary legal rules on rule 150 in the scenario of Fig.3.

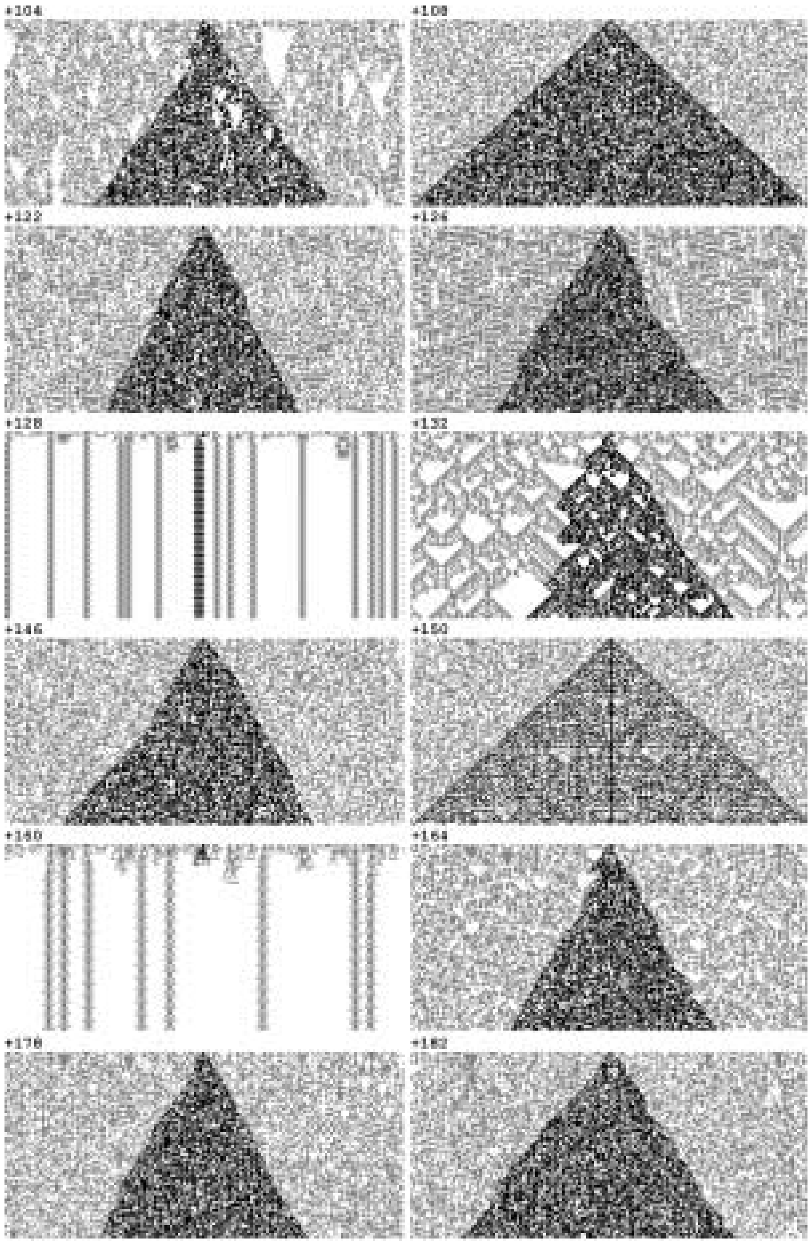


FIGURE 4
Countinued.

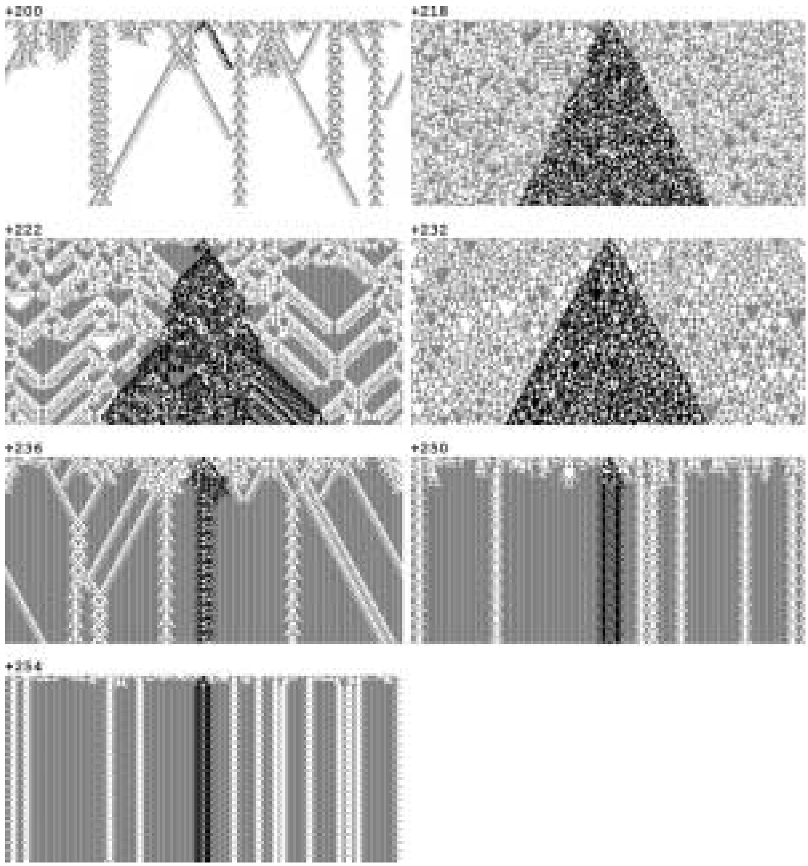


FIGURE 4
Countinued.

To the best of our knowledge the study of the effect of memory on CA has been rather neglected[5]. Some critics (an euphemism for referees) can argue that memory is not in the realm of CA (or even of Dynamic Systems), but we believe that the subject is worth studying. At least CA with memory can be considered as a promising extension of the basic paradigm. A major impediment in modeling with CA stems from the difficulty of utilizing the CA complex behavior to exhibit a particular behavior or perform a particular function: embedding memory in states broadens the spectrum of CA as a tool for modeling. It is likely that in some contexts, a transition rule with memory could match the "correct" behavior of the

[5] See, for example, Wolfram,S.([14],p.118), Ilachinski ([1],p.43), or class $\mathbb{CAM}$ in Adamatzky ([16],p.7). The latter shows an example of 2D CA with memory.

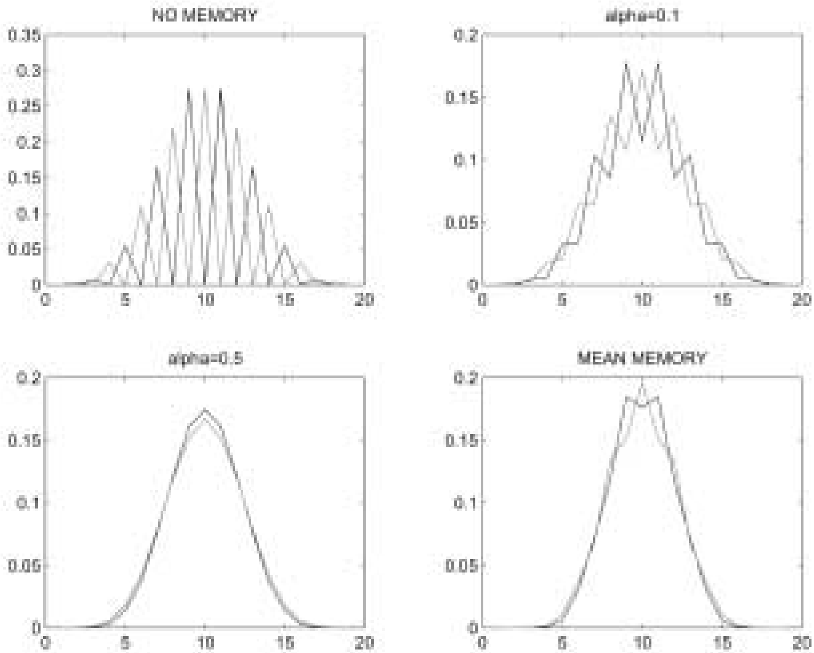


FIGURE 5
Random walk with memory. Probability distributions at $T = 9$ and at $T = 10$ starting in $X = 10$. Memory factor α.

CA system of a given complex system (physical, biological, social and so on).

The mechanism of implementation of memory adopted in this work, keeping the transition rule unaltered but appling it to a function of previous states, can be adopted in any dynamical system. In an earlier work ([3],[5]) we explored the effect of embedding this kind of memory into *discrete dynamical systems*: $x_{T+1} = f(x_T)$ by means of $x_{T+1} = f(m_T)$ with m_T being a mean value of past states. We have studied this approach in, perhaps, the canonical example: the logistic map: $x_{T+1} = x_T + \lambda x_T(1 - x_T)$, which becomes with memory $x_{T+1} = m_T + \lambda m_T(1 - m_T)$. In [4] we studied the effect of embedded memory on the two-dimensional Anosov map. Let us conclude this work by indicating the possibility of implementing embedded memory in Markovian stochastic processes, $\mathbf{p}'_{T+1} = \mathbf{p}'_T\mathbf{M}$ by means of $\mathbf{p}'_{T+1} = \boldsymbol{\pi}'_T\mathbf{M}$ with $\boldsymbol{\pi}_T = \pi(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_{T-1}, \mathbf{p}_T)$. A sort of *unconventional* Markov model with memory. As an example, $\boldsymbol{\pi}$ in Fig.5 is designed as a weighted mean of the last tree values: $\boldsymbol{\pi}_T(\mathbf{p}_{T-2}, \mathbf{p}_{T-1}, \mathbf{p}_T) = \frac{1}{\alpha^2+\alpha+1}(\alpha^2\mathbf{p}_{T-2} + \alpha\mathbf{p}_{T-1} + \mathbf{p}_T)$[6],

[6] With the initial conditions: $\boldsymbol{\pi}_1 = \mathbf{p}_1$, $\boldsymbol{\pi}_2(\mathbf{p}_1, \mathbf{p}_2) = \frac{1}{\alpha+1}(\alpha\mathbf{p}_1 + \mathbf{p}_2)$

with $0 \leq \alpha \leq 1$ actuating as a memory factor with extreme scenarios: the standard (ahistoric) model if $\alpha = 0$, and the arithmetic mean model $\boldsymbol{\pi}_T = \frac{1}{3}(\mathbf{p}_{T-2} + \mathbf{p}_{T-1} + \mathbf{p}_T)$ if $\alpha = 1$. Figure 5 shows the distribution of probability at $T = 9$ and at $T = 10$ in a symmetric ($p_{i-1,i} = p_{i,i+1} = 0.5$) random walk starting from $X = 10$. A minimal incorporation of memory ($\alpha = 0.1$) alters the characteristic alternating probabilistic distribution in the ahistoric model to a *serrated* form, much smoothed at higher α values as an effect of the mean memory implementation.

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