Characterization of the Reticulation of a Residuated Lattice

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In this article we prove several properties of the reticulation of a residuated lattice: the case of isomorphism between the residuated lattice and its reticulation, a universality property, a topological characterization and a series of functorial properties.

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1 INTRODUCTION

The reticulation of an algebra was first defined for commutative rings by Simmons [19] and it was extended by Belluce to non-commutative rings [3]. As for the algebras of fuzzy logics, Belluce also constructed the reticulation of an MV-algebra [2], G. Georgescu defined the reticulation of a quantale [8] and L. Leuştean made this construction for BL-algebras [13, 14].

In each of the papers cited above, although it is not explicitly defined this way, the reticulation of an algebra $A$ is a pair $(L(A), \lambda)$ consisting of a bounded distributive lattice $L(A)$ and a surjection $\lambda : A \to L(A)$ such that the function given by the inverse image of $\lambda$ induces (by restriction) a homeomorphism of topological spaces between the prime spectrum of $L(A)$ and that of $A$. This construction allows many properties to be transferred between $L(A)$ and $A$.

In [15] we defined the reticulation of a residuated lattice $A$ as a pair $(L(A), \lambda)$ consisting of a bounded distributive lattice $L(A)$ and a function $\lambda : A \to L(A)$ satisfying five conditions of algebraic nature, proved its

* Dedicated to my beloved parents, Gabriela and Florin-Alexandru
existence and uniqueness up to an isomorphism and defined the reticulation functor from the category of residuated lattices to the category of bounded distributive lattices.

In this article we recall the main definitions and results in [15] and then we continue the exploration of these mathematical concepts by proving several properties of the reticulation of a residuated lattice.

Except for some preparations in Section 2, this paper is composed entirely of original results. Also, the results in Section 2 that come with proofs are original as well, unless mentioned otherwise.

In Section 3 of this article we give a necessary and sufficient condition for a residuated lattice \( A \) to be isomorphic to the bounded distributive lattice \( L(A) \) from its reticulation \( (L(A), \lambda) \), with \( \lambda \) the isomorphism.

In Section 4 we prove a universality property (U) of the reticulation of a residuated lattice, give an equivalent definition of the reticulation using property (U) (Proposition 4.1) and show how the uniqueness of the reticulation can be proven using (U) (Theorem 4.1).

In Section 5 we give a topological characterization of the reticulation, by proving that, if \( A \) is a residuated lattice and \( (L(A), \lambda) \) is a pair consisting of a bounded distributive lattice \( L(A) \) and a surjection \( \lambda : A \rightarrow L(A) \) such that the function given by the inverse image of \( \lambda \) induces (by restriction) a homeomorphism of topological spaces between the prime spectrum of \( L(A) \) and that of \( A \), then this pair is indeed a reticulation of \( A \), i.e. the general notion of reticulation implies our definition of the reticulation of a residuated lattice. The converse implication being proven in Theorem 3.1 in [15], we thus get the equivalence between the two.

In Section 6 we give an alternate definition of the reticulation functor by using (U) and we prove several preservation properties of this functor, along with the fact that the reticulation functor for residuated lattices is not full.

2 PRELIMINARIES

For the basic definitions, notations and properties used in this article see [15]. We shall remind some of them here, together with some other known properties.

The following lemma is Theorem 2.3 from [6].

**Lemma 2.1.** A function between two lattices is a lattice isomorphism iff it is an order-preserving bijection whose inverse is also order-preserving.

**Definition 2.1.** Let \( L \) be a lattice. A nonempty subset \( F \) of \( L \) is called a filter of \( L \) iff it satisfies the following conditions:

(i) \( (\forall l, m \in F) l \land m \in F \);

(ii) \( (\forall l \in F)(\forall m \in L) \) if \( l \leq m \) then \( m \in F \).
The set of all filters of \( L \) is denoted \( \mathcal{F}(L) \).

A filter \( F \) of \( L \) is said to be proper iff \( F \neq L \).

**Definition 2.2.** Let \( L \) be a lattice. A proper filter \( P \) of \( L \) is called a *prime filter* iff, for all \( l, m \in L \), if \( l \lor m \in P \), then \( l \in P \) or \( m \in P \). The set of all prime filters of \( L \) is called the (prime) spectrum of \( L \) and is denoted \( \text{Spec}(L) \).

The following theorem is Corollary 1.3.11 on page 35 from [5].

**Theorem 2.1 (the prime filter theorem for distributive lattices).** Let \( L \) be a distributive lattice, \( F \) be a filter of \( L \) and \( l \in L \setminus F \). Then there exists a prime filter of \( L \) that includes \( F \) and does not contain \( l \).

**Definition 2.3.** A *residuated lattice* is an algebraic structure \((A, \lor, \land, \circ, \rightarrow, 0, 1)\), with the first 4 operations binary and the last two constant, such that \((A, \lor, \land, 0, 1)\) is a bounded lattice, \((A, \circ, 1)\) is a commutative monoid and the following property, called *residuation*, is satisfied: for all \( a, b, c \in A \), \( a \leq b \rightarrow c \iff a \circ b \leq c \), where \( \leq \) is the partial order of the lattice \((A, \lor, \land, 0, 1)\).

**Definition 2.4.** Let \( A \) be a residuated lattice. A nonempty subset \( F \) of \( A \) is called a *filter of \( A \)* iff it satisfies the following conditions:

(i) \((\forall a, b \in F) a \circ b \in F\);

(ii) \((\forall a \in F) (\forall b \in A) \text{ if } a \leq b \text{ then } b \in F\).

The set of all filters of \( A \) is denoted \( \mathcal{F}(A) \).

A filter \( F \) of \( A \) is said to be proper iff \( F \neq A \).

**Definition 2.5.** Let \( A \) be a residuated lattice. A proper filter \( P \) of \( A \) is called a *prime filter* iff, for all \( a, b \in A \), if \( a \lor b \in P \), then \( a \in P \) or \( b \in P \). The set of all prime filters of \( A \) is called the (prime) spectrum of \( A \) and is denoted \( \text{Spec}(A) \).

The following theorem is well known (its proof can be found, for example, in [16]).

**Theorem 2.2 (the prime filter theorem for residuated lattices).** Let \( A \) be a residuated lattice, \( F \) be a filter of \( A \) and \( a \in A \setminus F \). Then there exists a prime filter of \( A \) that includes \( F \) and does not contain \( a \).

At this point we note the similarity between Definitions 2.1, 2.2 and Theorem 2.1 on the one hand, and Definitions 2.4, 2.5 and Theorem 2.2 on the other hand. As a matter of fact, Theorems 2.1 and 2.2 can be stated jointly in a compact form:

**Theorem 2.3 (the prime filter theorem).** Let \( A \) be a distributive lattice (residuated lattice), \( F \) be a filter of \( A \) and \( a \in A \setminus F \). Then there exists a prime filter of \( A \) that includes \( F \) and does not contain \( a \).
As we are going to show below, Theorem 2.3 has several important consequences that can also be stated jointly.

**Corollary 2.1.** Any filter of a distributive lattice (residuated lattice) is equal to the intersection of the prime filters that include it.

**Definition 2.6.** Let \( A \) be a lattice (residuated lattice), \( X \subseteq A \) and \( a \in A \). The least filter of \( A \) that includes \( X \) (that is: the intersection of all filters of \( A \) that include \( X \)) is called the filter of \( A \) generated by \( X \) and is denoted by \( \langle X \rangle \).

The filter of \( A \) generated by \( \{a\} \) is denoted by \( \langle a \rangle \) and is called the principal filter of \( A \) generated by \( a \). For lattices, the notations mentioned above can be replaced by \( \{X\} \) and respectively \( \{a\} \).

**Notation 2.1.** Let \( A \) be a lattice (residuated lattice). For all filters \( F, G \) of \( A \), we denote \( < F \cup G > \) by \( F \lor G \). More generally, for any family \( \{F_t \mid t \in T\} \) of filters of \( A \), we denote \( \bigcup_{t \in T} F_t \) by \( \bigvee_{t \in T} F_t \).

**Notation 2.2.** Let \( A \) be a lattice (residuated lattice). For any \( X \subseteq A \), we shall denote \( D(X) = \{ P \in \text{Spec}(A) \mid X \subseteq P \} \). For any \( a \in A \), \( D(\{a\}) \) will be denoted \( D(a) \).

**Remark 2.1.** Let \( A \) be a distributive lattice (residuated lattice) and \( X_1, X_2 \) be subsets of \( A \). Then: if \( X_1 \subseteq X_2 \) then \( D(X_1) \subseteq D(X_2) \). If \( X_1, X_2 \) are filters of \( A \), then the converse implication is also true.

**Proof.** The first implication is obvious. If \( X_1 \) and \( X_2 \) are filters of \( A \) and \( D(X_1) \subseteq D(X_2) \), then, for all \( P \in \text{Spec}(A), X_2 \subseteq P \Rightarrow X_1 \subseteq P \), which, by Corollary 2.1, is equivalent to \( X_1 \subseteq X_2 \).

**Remark 2.2.** For any subset \( X \) of a lattice (residuated lattice) \( A \), \( D(X) = D(\langle X \rangle) \).

**Proof.** This is obvious, since any filter of \( A \) that includes \( X \) also includes \( \langle X \rangle \).

**Lemma 2.2.** Let \( A \) be a distributive lattice (residuated lattice) and \( \{X_t \mid t \in T\} \) be a family of subsets of \( A \). Then \( D(\bigcup_{t \in T} X_t) = \bigcup_{t \in T} D(X_t) \), hence, if the sets \( X_t \) are filters of \( A \), then \( D(\bigvee_{t \in T} X_t) = \bigvee_{t \in T} D(X_t) \).

**Proof.** If \( P \in D(\bigcup_{t \in T} X_t) \) then there exists \( t \in T \) such that \( X_t \subseteq P \), which is equivalent to \( P \in D(X_t) \subseteq \bigcup_{t \in T} D(X_t) \). Hence \( D(\bigcup_{t \in T} X_t) \subseteq \bigvee_{t \in T} D(X_t) \). The converse set inclusion holds by Remark 2.1. See Remark 2.2 for the second equality.

**Lemma 2.3.** Let \( A \) be a distributive lattice (residuated lattice) and \( X_1, X_2 \subseteq A \). Then: \( D(X_1) \subseteq D(X_2) \iff \langle X_1 \rangle \subseteq \langle X_2 \rangle \), and hence: \( D(X_1) = D(X_2) \iff \langle X_1 \rangle = \langle X_2 \rangle \).

**Proof.** It follows by Remarks 2.2 and 2.1 that: \( D(X_1) \subseteq D(X_2) \iff D(\langle X_1 \rangle) \subseteq D(\langle X_2 \rangle) \iff \langle X_1 \rangle \subseteq \langle X_2 \rangle \).
Lemma 2.4. Let \( L \) be a lattice and \( l \in L \). Then \( \{ l \} = \{ m \in L \mid l \leq m \} \).

The proof of the following remark can be found in [9, 12, 16, 20].

Remark 2.3. Let \( A \) be a residuated lattice and \( a, b, c, d \in A \). Then:

(i) if \( a \leq b \), then \( a \circ c \leq b \circ c \);
(ii) if \( a \leq b \) and \( c \leq d \), then \( a \circ c \leq b \circ d \).

Notation 2.3. Let \( A \) be a residuated lattice, \( a \in A \) and \( n \in \mathbb{N}^* \). We shall denote by \( a^n \) the following element of \( A \): \( a \circ \ldots \circ a \). We also denote \( a^0 = 1 \).

Lemma 2.5. Let \( A \) be a residuated lattice and \( a \in A \). Then \( \langle a \rangle = \{ b \in A \mid (\exists n \in \mathbb{N}^*) a^n \leq b \} \).

This lemma has been proven in [15]. It has the following obvious corollary (see Remark 2.3).

Corollary 2.2. Let \( A \) be a residuated lattice and \( a, b \in A \) such that \( a \leq b \). Then \( \langle a \rangle \supseteq \langle b \rangle \).

Lemma 2.5 could have been obtained as a corollary of the following lemma.

Lemma 2.6. Let \( A \) be a residuated lattice and \( X \subseteq A \). Then \( \langle X \rangle = \{ 1 \} \) if \( X = \emptyset \) and \( \langle X \rangle = \{ a \in A \mid (\exists n \in \mathbb{N}^*) (\exists a_1, \ldots, a_n \in X) a_1 \circ \ldots \circ a_n \leq a \} \) otherwise.

Proof. The fact that \( \langle \emptyset \rangle = \{ 1 \} \) is easily seen from Definition 2.6. Let us assume that \( X \) is nonempty and let \( F \) be the set in the right hand side of the second equality in the enunciation. It is obvious by Definition 2.4 that \( F \subseteq \langle X \rangle \). For proving the converse set inclusion, let us first notice that \( X \subseteq F \); hence it remains to show that \( F \) is a filter. \( F \) is nonempty, since it includes \( X \), and it is easily seen that it also satisfies conditions (i) and (ii) from Definition 2.4 (for proving condition (i) apply property (ii) from Remark 2.3).

Remark 2.4. Let \( A \) be a residuated lattice and \( a, b \in A \) such that \( a \leq b \). Then \( D(a) \supseteq D(b) \).

Proof. By Corollary 2.2 and Lemma 2.3.

The equivalents of the following two remarks for BL-algebras are Proposition 1.6.3, (vii) and respectively (vi), from [14] (also found in [13]); their proofs are also valid for residuated lattices, and, in the case of (vi), also for lattices.

Remark 2.5. Let \( A \) be a residuated lattices and \( a, b \in A \). Then: \( D(a \circ b) = D(a \wedge b) = D(a) \cup D(b) \).
Remark 2.6. Let $A$ be a lattice (residuated lattice) and $a, b \in A$. Then: $D(a \vee b) = D(a) \cap D(b)$.

The equivalent of the following proposition for BL-algebras can be found in [13, 14].

Proposition 2.1. For any distributive lattice (residuated lattice) $A$, the family $\{D(X) \mid X \subseteq A\}$ is a topology on $\text{Spec}(A)$, having the basis $\{D(a) \mid a \in A\}$.

Proof. Lemma 2.2 shows that, for all subsets $X$ of $A$, $D(X) = \bigcup_{a \in X} D(a)$, hence it remains to prove that $\{D(X) \mid X \subseteq A\}$ is a topology. By Lemma 2.2, this family of sets is closed with respect to arbitrary unions. The equality above and Remark 2.6 show that, for all subsets $X, Y$ of $A$:

$$D(X) \cap D(Y) = \left( \bigcup_{a \in X} D(a) \right) \cap \left( \bigcup_{b \in Y} D(b) \right) = \bigcup_{a \in X, b \in Y} (D(a) \cap D(b))$$

$$= \bigcup_{a \in X, b \in Y} D(a \vee b) = D(\{a \vee b \mid a \in X, b \in Y\}).$$

The properties $D(\emptyset) = \emptyset$ and $D(A) = \text{Spec}(A)$ are obvious. □

Definition 2.7. For any distributive lattice (residuated lattice) $A$, the family $\{D(X) \mid X \subseteq A\}$ is called the Stone topology of $A$.

Remark 2.7. The Stone topology of a lattice (residuated lattice) $A$ is equal to $\{D(F) \mid F \in \mathcal{F}(A)\}$.

Proof. By Remark 2.2. □

Notation 2.4. For any set $X$, let us denote by $\mathcal{P}(X)$ the set of the subsets of $X$. For any function $f : X \to Y$, we shall denote by $f^* : \mathcal{P}(Y) \to \mathcal{P}(X)$ the function given by its inverse image: $(\forall M \in \mathcal{P}(Y)) f^*(M) = f^{-1}(M) = \{x \in X \mid f(x) \in M\}$.

The results below concerning the functor $K$ can be found in [5].

Definition 2.8. A $T_0$ space is a topological space $X$ that fulfills the $T_0$-separation axiom: for any $x, y \in X$, there exists an open set $V$ such that: $x \notin V$ and $y \in V$, or $x \in V$ and $y \notin V$.

Notation 2.5. If $X$ is a topological space, then by $KX$ or $K(X)$ we denote the set of the compact open subsets of $X$.

Definition 2.9. A Stone space is a topological space $X$ that satisfies the following conditions:

(i) $X$ is a compact $T_0$ space;
(ii) $K_X$ is a ring of sets (i.e. closed with respect to finite intersections and finite unions) and a basis for the topology of $X$;

(iii) if $C \subseteq K_X$ is closed with respect to finite intersections and $F \subseteq X$ is a closed set such that, for every $Y \in C$, $F \cap Y \neq \emptyset$, then $F \cap \bigcap_{Y \in C} Y \neq \emptyset$.

The following proposition is (1.24) of Lemma 2.1.13 in [5].

**Proposition 2.2.** For any Stone space $X$, $(K_X, \cup, \cap, \emptyset, X)$ is a bounded distributive lattice.

**Definition 2.10.** A mapping $f : X \to Y$ between two Stone spaces $X$ and $Y$ is said to be strongly continuous iff, for every $A \in KY$, $f^*(A) \in K_X$.

**Notation 2.6.** We denote by $ST$ the category of Stone spaces and strongly continuous functions.

**Notation 2.7.** We denote by $D_{01}$ the category of bounded distributive lattices.

**Notation 2.8.** For any Stone spaces $X, Y$, and for any $f \in ST(X, Y)$, set $Kf : KY \to KX$, defined by: for all $A \in KY$, $Kf(A) = f^*(A)$.

**Lemma 2.7.** The functor $K : ST \to D_{01}$ described in Notations 2.5 and 2.8 is well defined and it is a categorical duality, whose inverse is the functor $\text{Spec} : D_{01} \to ST$, defined by: for any bounded distributive lattice $L$, $\text{Spec}(L)$ is the prime spectrum of $L$, and, for any bounded lattice morphism $f : L \to L_1$, $\text{Spec}(f) \in ST(\text{Spec}(L_1), \text{Spec}(L))$, for all $P \in \text{Spec}(L_1)$, $\text{Spec}(f)(P) = f^*(P)$.

**Proposition 2.3.** Let $L$ be a bounded distributive lattice. Then $K(\text{Spec}(L)) = \{D(l) \mid l \in L\}$.

**Proof.** By duality, from relation (1.13) of Theorem 2.1.8 on page 85 from [5].

**Proposition 2.4.** Let $A$ be a residuated lattice. Then $K(\text{Spec}(A)) = \{D(a) \mid a \in A\}$.

**Proof.** Let us first prove that $K(\text{Spec}(A)) \supseteq \{D(a) \mid a \in A\}$. Let $a \in A$. We know that $D(a)$ is open (see Definition 2.7). In order to prove that it is also compact, we shall use Remark 2.7. Let $\{F_t \mid t \in T\}$ be a family of filters of $A$ such that $D(a) \subseteq \bigcup_{t \in T} D(F_t)$. By Lemma 2.2, this is equivalent to: $D(a) \subseteq D(\bigvee_{t \in T} F_t)$, that is (by Remarks 2.1 and 2.2): $a \in \bigvee_{t \in T} F_t$, which in turn (by Notation 2.1 and Lemma 2.6) is equivalent to: $(\exists t_1, \ldots, t_n \in T)(\exists a_1 \in F_{t_1}) \cdots (\exists a_n \in F_{t_n}) a_1 \circ \cdots \circ a_n \leq a$. This, by Remark 2.4, implies that $D(a) \subseteq D(a_1 \circ \cdots \circ a_n) = D(a_1) \cup \cdots \cup D(a_n)$. For the last equality see Remark 2.5. Now, using Remark 2.1, we see that, for all $i \in \overline{1, n}$,
$D(a_i) \subseteq D(F_i)$, therefore $D(a) \subseteq D(F_1) \cup \cdots \cup D(F_n)$. This proves that $D(a)$ is compact, hence the desired set inclusion.

Let us now prove the converse set inclusion, namely that any open and compact subset of $\text{Spec}(A)$ is of the form $D(a)$, with $a \in A$. Let $X \subseteq A$, such that $D(X) \in K(\text{Spec}(A))$. From Lemma 2.2, we get that $D(X) = \bigcup_{a \in X} D(a)$. Since $D(X)$ is compact, this implies that there exist $a_1, \ldots, a_n \in X$ such that $D(X) = \bigcup_{i=1}^n D(a_i) = D(a_1 \circ \cdots \circ a_n)$, by Remark 2.5. So we have proven the converse set inclusion and hence the equality in the enunciation.

\begin{proof}
\end{proof}

\begin{definition}
A covariant functor $F : C \to D$ is said to be \textit{full} iff, for any objects $A, B$ of $C$, the map $F : C(A, B) \to D(F(A), F(B))$ is surjective.
\end{definition}

\begin{definition}
Let $F : C \to D$ be a functor and $P$ a property which makes sense for the morphisms of $C$ and those of $D$. The functor $F$ \textit{preserves} the property $P$ iff, for any morphism $f$ of the category $C$ that verifies $P$, $F(f)$ also verifies $P$. The functor $F$ \textit{reflects} the property $P$ iff, for any morphism $f$ of the category $C$, if $F(f)$ satisfies $P$, then $f$ satisfies $P$.
\end{definition}

The following remark is well known.

\begin{remark}
Any functor preserves isomorphisms.
\end{remark}

\begin{proof}
Let $F : C \to D$ be a functor, $A$ and $B$ objects in $C$, $f : A \to B$ an isomorphism in $C$ and $g : B \to A$ its inverse. Then $f \circ g = \text{id}_B$ and $g \circ f = \text{id}_A$. Applying $F$ we get: $F(f) \circ F(g) = id_{F(B)}$ and $F(g) \circ F(f) = id_{F(A)}$, so $F(f)$ is an isomorphism in $D$.
\end{proof}

The following lemma can be found in [17] (Theorem 2.2.12 on page 62) or, in a more general version, in [18] (Proposition 1.14 on page 18). For the terminology we are using here see [6].

\begin{lemma}
Let $A, B, C$ be universal algebras of the same type and $\phi : A \to B$, $\psi : A \to C$ be morphisms such that $\phi$ is surjective. Then $\text{Ker}\phi \subseteq \text{Ker}\psi$ iff there exists a unique morphism $\theta : B \to C$ such that $\theta \circ \phi = \psi$. Moreover, $\theta$ is surjective iff $\psi$ is surjective.
\end{lemma}

\begin{proof}
First let us prove the direct implication.

For proving the existence of $\theta$, let us define $\theta : B \to C$ by: for all $a \in A$, $\theta(\phi(a)) = \psi(a)$. Since $\phi$ is surjective, $\theta$ is completely defined. Let $a_1, a_2 \in A$ such that $\phi(a_1) = \phi(a_2)$, i.e. $(a_1, a_2) \in \text{Ker}\phi$, so $(a_1, a_2) \in \text{Ker}\psi$,

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (B) at (1,0) {$B$};
  \node (C) at (1,-1) {$C$};
  \draw[->] (A) to (B);
  \draw[->] (A) to (C);
  \draw[->] (B) to (C);
  \node at (0.5, -0.5) {$\phi$};
  \node at (0.5, -0.8) {$\psi$};
  \node at (1.2, -0.5) {$\theta$};
\end{tikzpicture}
\end{center}

\begin{proof}
First let us prove the direct implication.

For proving the existence of $\theta$, let us define $\theta : B \to C$ by: for all $a \in A$, $\theta(\phi(a)) = \psi(a)$. Since $\phi$ is surjective, $\theta$ is completely defined. Let $a_1, a_2 \in A$ such that $\phi(a_1) = \phi(a_2)$, i.e. $(a_1, a_2) \in \text{Ker}\phi$, so $(a_1, a_2) \in \text{Ker}\psi$,
i.e. $\psi(a_1) = \psi(a_2)$, which is equivalent to $\theta(\phi(a_1)) = \theta(\phi(a_2))$. So $\theta$ is well defined.

Let $\sigma$ be a function symbol from the type of $A$, of arity $n$, and let $\sigma_A$, $\sigma_B$, $\sigma_C$ be the operations of $A$, $B$ and $C$ respectively corresponding to $\sigma$. If $n = 0$, then $\theta(\sigma_B) = \theta(\phi(\sigma_A)) = \psi(\sigma_A) = \sigma_C$. If $n \geq 1$, let $b_1, \ldots, b_n \in B$. The surjectivity of $\phi$ implies that there exist $a_1, \ldots, a_n \in A$ such that, for all $k \in \mathbb{N}$, $\phi(a_k) = b_k$. The following equalities hold, because $\phi$ and $\psi$ are morphisms and because of the definition of $\theta$: $\theta(\psi(a_1), \ldots, \psi(a_n)) = \sigma_C(\theta(\phi(a_1)), \ldots, \theta(\phi(a_n))) = \sigma_C(\theta(b_1), \ldots, \theta(b_n))$. Hence $\theta$ is a morphism.

For showing the uniqueness of $\theta$, let $\theta_1 : B \to C$ be a morphism such that $\theta_1 \circ \phi = \psi$. Let $b \in B$. $\phi$ is surjective, so there exists $a \in A$ such that $\phi(a) = b$. Therefore $\theta_1(b) = \theta_1(\phi(a)) = \psi(a)$. So $\theta$ is unique.

For the converse implication, let $(a_1, a_2) \in \text{Ker} \phi$, i.e. $\phi(a_1) = \phi(a_2)$, hence $\theta(\phi(a_1)) = \theta(\phi(a_2))$, which is equivalent to $\psi(a_1) = \psi(a_2)$. So $\text{Ker} \phi \subseteq \text{Ker} \psi$.

The definition of $\theta$ and the surjectivity of $\phi$ obviously imply the equivalence between the surjectivity of $\theta$ and that of $\psi$. $\square$

In [15] we gave the following definition of the reticulation of a residuated lattice and established the lemma below.

**Definition 2.13.** Let $A$ be a residuated lattice. A reticulation of $A$ is a pair $(L, \lambda)$, where $L$ is a bounded distributive lattice and $\lambda : A \to L$ is a function that satisfies conditions (1)–(5) below:

1. $(\forall a, b \in A) \lambda(a \circ b) = \lambda(a) \wedge \lambda(b)$;
2. $(\forall a, b \in A) \lambda(a \vee b) = \lambda(a) \vee \lambda(b)$;
3. $\lambda(0) = 0$; $\lambda(1) = 1$;
4. $\lambda$ is surjective;
5. for all $a, b \in A$, $\lambda(a) \leq \lambda(b)$ iff $(\exists n \in \mathbb{N}^+) a^n \leq b$.

**Lemma 2.9.** If $A$ is a residuated lattice and $L$ is a bounded distributive lattice, then a function $\lambda : A \to L$ that verifies conditions (1)–(3) also satisfies:

(a) $\lambda$ is order-preserving;
(b) $(\forall a, b \in A) \lambda(a \wedge b) = \lambda(a) \wedge \lambda(b)$;
(c) $(\forall a \in A)(\forall n \in \mathbb{N}^+) \lambda(a^n) = \lambda(a)$.

We shall use the notations of the conditions (1)–(5) and of the properties (a)–(c) in the following sections also.
Notation 2.9. We denote by $\mathcal{RL}$ the category of residuated lattices.

In [15] we defined the reticulation functor by using the first construction of the reticulation (see below). But all reticulations of a residuated lattice are isomorphic (see Theorem 4.3 in [15] and Theorem 4.1 in this paper), so we can give a definition of the reticulation that is independent of its construction and then apply this definition to the two constructions of the reticulation that we gave in [15]. Here is this definition of the reticulation functor $\mathcal{L}$.

Definition 2.14. Let $A$ be a residuated lattice and $(L(A), \lambda_A)$ be its reticulation. Then $\mathcal{L}(A) = L(A)$.

Let $B$ be another residuated lattice, $(L(B), \lambda_B)$ be its reticulation and $f : A \to B$ be a morphism of residuated lattices. Then $\mathcal{L}(f) : L(A) \to L(B) = L(B)$, for all $a \in A$, $\mathcal{L}(f)(\lambda_A(a)) = \lambda_B(f(a))$.

With this definition $\mathcal{L}$ is a covariant functor from $\mathcal{RL}$ to $\mathcal{D}_{01}$ (see Proposition 4.5 in [15] for the first construction of the reticulation, that we show below, and see Proposition 6.1 in the present article for this general construction).

Let $A$ be a residuated lattice. Here are the two constructions of the reticulation that we gave in [15], together with the definitions of the reticulation functor that one obtains from Definition 2.14 for these particular cases.

The first construction in [15] is based on the following binary relation on $A$: for all $a, b \in A$, $a \equiv b$ iff $D(a) = D(b)$. For all $a \in A$, we denote by $[a]$ the equivalence class of $a$ and by $A/\equiv$ the quotient set of $A$ with respect to the equivalence relation $\equiv$. We proved in [15] that $\equiv$ is a congruence relation of the algebra $(A, \lor, \land, \cdot, 0, 1)$ (see Proposition 4.1 in [15]). According to a standard construction, we define on the quotient set $A/\equiv$ the following operations: $\lor, \land, 0$ and $1$, by:

$$\begin{align*}
(\forall a, b \in A) \ [a] \lor [b] &= [a \lor b], \\
(\forall a, b \in A) \ [a] \land [b] &= [a \land b] = [a \land b], \\
0 &= [0], \ 1 = [1]
\end{align*}$$

and let $\lambda : A \to A/\equiv$ be the canonical surjection. According to Theorem 4.1 in [15], $(A/\equiv, \lambda)$ is a reticulation of $A$.

The definition of functor $\mathcal{L}$ using this construction is obvious: $\mathcal{L}(A) = A/\equiv$ and, if $f : A \to B$ is a morphism of residuated lattices, then, for all $a \in A$, $\mathcal{L}(f)([a]) = [f(a)]$.

The second construction of the reticulation uses principal filters. In [15] we denoted by $\mathcal{P}\mathcal{F}(A)$ the set of principal filters of $A$ and by $\lambda : A \to \mathcal{P}\mathcal{F}(A)$ the following function: for all $a \in A$, $\lambda(a) = \langle a \rangle$. According to Theorem 4.2 in [15], $((\mathcal{P}\mathcal{F}(A), \cap, \lor, A, \{1\}), \lambda)$ is a reticulation of $A$.

Again, the definition of functor $\mathcal{L}$ using this construction is obvious: $\mathcal{L}(A) = \mathcal{P}\mathcal{F}(A)$ and, if $f : A \to B$ is a morphism of residuated lattices, then, for all $a \in A$, $\mathcal{L}(f)(\langle a \rangle) = \langle f(a) \rangle$. 
3 RESIDUATED LATTICES WHICH ARE ISOMORPHIC TO THEIR RETICULATION

Proposition 3.1. Let $A$ be a residuated lattice and $(L, \lambda)$ be its reticulation. Then the following assertions are equivalent:

(i) $\lambda$ is a bounded lattice isomorphism;
(ii) $\odot = \land$;
(iii) all elements of $A$ are idempotent.

Proof. The fact that all elements of $A$ are idempotent iff $\odot = \land$ is known (see [16], Proposition 1.20 and its proof). However, we will give a new proof of this equivalence here.

Let $(i')$ $\lambda$ is an injection.

$\lambda$ is always a surjective bounded lattice morphism, by Definition 2.13 and Lemma 2.9. Hence what we have to show is that $(i')$ implies (ii) implies (iii) implies $(i')$.

If $\lambda$ is an injection, then since for every $a, b \in A$ we have $\lambda(a \odot b) = \lambda(a \land b)$ by Definition 2.13(1) and Lemma 2.9(b), it follows that $a \odot b = a \land b$. Hence $(i')$ implies (ii).

It is trivial that (ii) implies (iii).

If all elements of $A$ are idempotent, then condition (5) in Definition 2.13 becomes: for all $a, b \in A$, $\lambda(a) \leq \lambda(b)$ iff $a \leq b$. This implies: for all $a, b \in A$, $\lambda(a) = \lambda(b)$ iff $a = b$. Hence $\lambda$ is injective. So (iii) implies $(i')$ and the equivalence in the enunciation is now proved. $\square$

4 A UNIVERSALITY PROPERTY OF THE RETICULATION

In this section we will prove a universality property of the reticulation of a residuated lattice and use this property for giving an equivalent definition of the reticulation and for proving the uniqueness of the reticulation.

Definition 4.1. Let $A$ be a residuated lattice and $(L, \lambda)$ be a pair consisting of a bounded distributive lattice $L$ and a function $\lambda : A \rightarrow L$. We say that $\lambda$ satisfies the universality property $(U)$ iff, for all pairs $(L_1, f)$ consisting of a bounded distributive lattice $L_1$ and a function $f : A \rightarrow L_1$ that satisfies conditions (1)–(3), there exists a unique morphism of bounded lattices $\overline{f} : L \rightarrow L_1$ such that $\overline{f} \circ \lambda = f$ (i.e. that makes the diagram below commutative).

\[
\begin{array}{ccc}
A & \xrightarrow{\lambda} & L \\
\downarrow{f} & & \downarrow{\overline{f}} \\
L_1 & & 
\end{array}
\]
The following result establishes the equivalence between condition (5) from the definition of the reticulation and the universality property (U), provided that conditions (1)–(4) are satisfied.

**Proposition 4.1.** Let $A$ be a residuated lattice, $L$ a bounded distributive lattice and $\lambda : A \to L$ a function that verifies conditions (1)–(4). Then: $\lambda$ satisfies condition (5) (thus $(L, \lambda)$ is a reticulation of $A$) iff it satisfies the universality property (U).

**Proof.**

Let us first prove that (5) implies (U). So let us assume that $\lambda$ verifies (1)–(5).

Let us consider a bounded distributive lattice $L_1$ and a function $f : A \to L_1$ that verifies conditions (1)–(3) (see the diagram in Definition 4.1). Lemma 2.9(b), combined with (2) and (3), shows that both $\lambda$ and $f$ are bounded lattice morphisms. In addition to that, $\lambda$ is surjective, by (4). In view of Lemma 2.8, to prove (U) it suffices to show that $\text{Ker} \, \lambda \subseteq \text{Ker} \, f$. Let $(a, b) \in \text{Ker} \, \lambda$, i.e. $\lambda(a) = \lambda(b)$. We have the following set of equivalences: $\lambda(a) = \lambda(b)$ iff $\lambda(a) \leq \lambda(b)$ and $\lambda(b) \leq \lambda(a)$ iff (by (5)) $a^n \leq b$ and $b^k \leq a$ for some $n, k \in \mathbb{N}_*$. In view of (a) and (c) on $f$, it follows that $f(a) \leq f(b)$ and $f(b) \leq f(a)$, hence $f(a) = f(b)$. So $(a, b) \in \text{Ker} \, f$, therefore (U) is satisfied.

And now conversely: (U) implies (5). We shall assume that $\lambda$ verifies (1)–(4) and (U).

Let $(L_1, f)$ be a reticulation of $A$. Such a pair exists (see [15], Theorems 4.1 and 4.2). By (U), there exists a unique bounded lattice morphism $\overline{f} : L \to L_1$ such that $\overline{f} \circ \lambda = f$. Let $a, b \in A$. We must prove the equivalence in (5) for $\lambda$ and these elements of $A$. So first let us assume that $\lambda(a) \leq \lambda(b)$. We have the following set of equivalences: $\lambda(a) = \lambda(b)$ iff $\lambda(a) \leq \lambda(b)$ and $\lambda(b) \leq \lambda(a)$ iff (by (5)) $a^n \leq b$ and $b^k \leq a$ for some $n, k \in \mathbb{N}_*$. In view of (a) and (c) on $f$, it follows that $f(a) \leq f(b)$ and $f(b) \leq f(a)$, hence $f(a) = f(b)$. So $(a, b) \in \text{Ker} \, f$, therefore (U) is satisfied.

And now conversely: (U) implies (5). We shall assume that $\lambda$ verifies (1)–(4) and (U).

Let us now prove the uniqueness of the reticulation by using the universality property (U) and without using condition (5).

**Theorem 4.1.** Let $A$ be a residuated lattice, $L$ and $L_1$ be bounded distributive lattices and $\lambda : A \to L$, $\lambda_1 : A \to L_1$ be functions that verify properties (1)–(4) and (U).

Then there exists a unique bounded lattice isomorphism $h : L \to L_1$ such that $h \circ \lambda = \lambda_1$.
Proof. Since $\lambda$ satisfies (U) and $\lambda_1$ verifies (1)–(3), by the definition of (U) we get that there exists a unique bounded lattice morphism $h : L \to L_1$ such that $h \circ \lambda = \lambda_1$. Similarly, the fact that $\lambda_1$ satisfies (U) and $\lambda$ verifies (1)–(3) implies that there exists a unique bounded lattice morphism $h_1 : L_1 \to L$ such that $h_1 \circ \lambda_1 = \lambda$. Therefore $h \circ h_1 \circ \lambda_1 = \lambda_1$ and $h_1 \circ h \circ \lambda = \lambda$. Since $\lambda$ and $\lambda_1$ are surjections, it follows that $h \circ h_1 = id_{L_1}$ and $h_1 \circ h = id_L$, which concludes the proof. 

We shall use the notation of the universality property (U) in the following sections also.

5 A TOPOLOGICAL CHARACTERIZATION OF THE RETICULATION OF A RESIDUATED LATTICE

In this section we prove that the general notion of reticulation implies our definition of the reticulation of a residuated lattice, by showing that, if $A$ is a residuated lattice and $(L(A), \lambda)$ is a pair consisting of a bounded distributive lattice $L(A)$ and a surjection $\lambda : A \to L(A)$ such that the function given by the inverse image of $\lambda$ induces (by restriction) a homeomorphism of topological spaces between the prime spectrum of $L(A)$ and that of $A$, then this pair is indeed a reticulation of $A$, that is it satisfies conditions (1)–(5). See Theorem 3.1 in [15] for the converse implication. This shows that, in fact, our definition of the reticulation of a residuated lattice is equivalent to the general notion of reticulation applied to residuated lattices.

Lemma 5.1. Let $A$ be a residuated lattice, $L$, $L_1$ be bounded distributive lattices, $\lambda : A \to L$ be a function and $\phi : L \to L_1$ be a bounded lattice isomorphism. Then: $\lambda$ verifies condition (1) iff $\phi \circ \lambda$ verifies condition (1) and the same is valid for each of the conditions (2)–(5).

\[
\begin{array}{ccc}
A & \xrightarrow{\lambda} & L \\
\downarrow{\phi} & & \downarrow{\phi} \\
L_1 & & L_1
\end{array}
\]

Proof. For condition (1), let $a, b \in A$. Then: $(\phi \circ \lambda)(a \circ b) = (\phi \circ \lambda)(a) \land (\phi \circ \lambda)(b)$ iff $\phi(\lambda(a \circ b)) = \phi(\lambda(a)) \land \phi(\lambda(b))$ iff (because $\phi$ is a bounded lattice morphism) $\phi(\lambda(a \circ b)) = \phi(\lambda(a) \land \lambda(b))$ iff (since $\phi$ is injective) $\lambda(a \circ b) = \lambda(a) \land \lambda(b)$.

A similar set of equivalences proves the affirmation for condition (2).

For condition (3), we have: $\phi(\lambda(0)) = 0 \iff \lambda(0) = 0$ and $\phi(\lambda(1)) = 1 \iff \lambda(1) = 1$, because $\phi$ commutes with 0 and 1 and it is injective.

Let us prove the affirmation for condition (4). If $\lambda$ is surjective, then, since $\phi$ is surjective, we get that $\phi \circ \lambda$ is surjective. Conversely, if $\phi \circ \lambda$ is surjective,
then for every $l \in L$ there exists $a \in A$ such that $\phi(l) = (\phi \circ \lambda)(a)$, which is equivalent to $l = \lambda(a)$, since $\phi$ is injective; so $\lambda$ is surjective.

For condition (5), let $a, b \in A$. It is sufficient to notice that, since $\phi$ is a lattice isomorphism, $\phi(\lambda(a)) \leq \phi(\lambda(b)) \iff \lambda(a) \leq \lambda(b)$ (see Lemma 2.1).

**Proposition 5.1.** Let $A$ be a residuated lattice, $L$ a bounded distributive lattice and $\lambda : A \to L$ a function with the following properties:

(I) for all $P \in \text{Spec}(L)$, $\lambda^*(P) \in \text{Spec}(A)$;

(II) $\lambda^* : \text{Spec}(L) \to \text{Spec}(A)$ is a homeomorphism of topological spaces.

Then $(L, \lambda)$ is a reticulation of $A$.

**Proof.** Let us consider the functor $K : \mathcal{ST} \to \mathcal{D}_01$ (see Notations 2.5 and 2.8) and the restriction $\lambda^* : \text{Spec}(L) \to \text{Spec}(A)$ of the function $\lambda^*$, which we shall also denote by $\lambda^*$.

Applying $K$ to the homeomorphism $\lambda^* : \text{Spec}(L) \to \text{Spec}(A)$ we get the following bounded lattice isomorphism (see Remark 2.8): $K(\lambda^*) : K(\text{Spec}(A)) \to K(\text{Spec}(L)), (\forall U \in K(\text{Spec}(A))) K(\lambda^*)(U) = (\lambda^*)(U) = \{P \in \text{Spec}(L) | \lambda^*(P) \in U\}$.

Let $\mu : A \to K(\text{Spec}(A))$, $(\forall a \in A) \mu(a) = D(a) = \{P \in \text{Spec}(A) | a \neq P\}$. According to Proposition 2.4, $\mu$ is well defined.

Let $\phi : L \to K(\text{Spec}(L))$, $(\forall l \in L) \phi(l) = D(l) = \{P \in \text{Spec}(L) | l \neq P\}$. According to Proposition 2.3, $\phi$ is well defined and surjective. Let us prove that $\phi$ is a bounded lattice isomorphism between $(L, \leq, 0, 1)$ and $(K(\text{Spec}(L)), \supseteq, \text{Spec}(L), \emptyset)$. Let $l, m \in L$, such that $l \leq m$. This (by Lemmas 2.4 and 2.3) is equivalent to $D(l) \supseteq D(m)$, which in turn is equivalent to $\phi(l) \supseteq \phi(m)$. The equivalence we have just proven shows that: $l = m$ iff $\phi(l) = \phi(m)$. So $\phi$ is injective. Hence $\phi$ is an order-preserving bijection whose inverse is also order-preserving, thus it is a lattice isomorphism, as Lemma 2.1 shows. $\phi(0) = \{P \in \text{Spec}(L) | 0 \neq P\} = \{P \in \text{Spec}(L) | P \neq L\} = \text{Spec}(L)$, according to Definition 2.2. $\phi(1) = \{P \in \text{Spec}(L) | 1 \notin P\} = \emptyset$. So $\phi$ is a bounded lattice isomorphism.

Let us prove that $K(\lambda^*) \circ \mu = \phi \circ \lambda$, i.e. that the diagram below is commutative.

```
A -------- L
\ |        \|
\mu \downarrow \phi
V K(\text{Spec}(A)) \rightarrow K(\text{Spec}(L))
```

Let $a \in A$. $K(\lambda^*)(\mu(a)) = \{P \in \text{Spec}(L) | \lambda^*(P) \in \mu(a)\} = \{P \in \text{Spec}(L) | \lambda^*(P) \in D(a)\} = \{P \in \text{Spec}(L) | a \notin \lambda^*(P)\} = \{P \in \text{Spec}(L) | \lambda(a) \notin P\} = D(\lambda(a)) = \phi(\lambda(a))$. Therefore $K(\lambda^*) \circ \mu = \phi \circ \lambda$. 
Let us prove that \( \mu \) verifies conditions (1)–(5), as a mapping from \( A \) to the bounded distributive lattice \( (K(\text{Spec}(A)), \cap, \cup, \text{Spec}(A), \emptyset) \) (see Proposition 2.2), with the partial order relation given by \( \supseteq \). Remark 2.5 proves that \( \mu \) satisfies condition (1) and Remark 2.6 proves that it verifies (2). \( \mu(0) = \{ P \in \text{Spec}(A) \mid 0 \notin P \} = \{ P \in \text{Spec}(A) \mid P \neq A \} = \text{Spec}(A) \), according to Definition 2.5. \( \mu(1) = \{ P \in \text{Spec}(A) \mid 1 \notin P \} = \emptyset \). So \( \mu \) verifies (3). From Proposition 2.4, we get that \( \mu \) is surjective, so it satisfies (4). For proving (5), let \( a, b \in A \). By using in turn Lemma 2.5, Definition 2.6 and Lemma 2.3, we get: \( (\exists n \in \mathbb{N}^+) \ a^n \leq b \iff b \in \langle a \rangle \iff b \subseteq \langle a \rangle \iff D(b) \subseteq D(a) \iff \mu(b) \subseteq \mu(a) \iff \mu(a) \supseteq \mu(b) \).

So \( \mu \) verifies conditions (1)–(5). By using in turn Lemma 5.1, the equality \( K(\lambda^*) \circ \mu = \phi \circ \lambda \), again Lemma 5.1 and the definition of the reticulation, we obtain the following set of equivalences, that proves the enunciation: \( \mu \) verifies conditions (1)–(5) iff \( K(\lambda^*) \circ \mu \) verifies conditions (1)–(5) iff \( \phi \circ \lambda \) verifies conditions (1)–(5) iff \( \lambda \) verifies conditions (1)–(5) iff \( (L, \lambda) \) is a reticulation of \( A \).

6 FUNCTIONAL PROPERTIES OF THE RETICULATION

We remind the reader that in [15] the reticulation functor was defined by using the first construction of the reticulation. In Section 2 we gave a definition for this functor that is independent of the construction of the reticulation. The action of this functor on morphisms can be described in a more direct manner by using the universality property (U); this way a more natural definition of the reticulation functor is obtained. This is the first objective of this section. The other objective is to prove several properties of the reticulation functor.

So let us now use the universality property (U) to define the reticulation functor on morphisms. We recall that, if \( A \) is a residuated lattice, then \( \mathcal{L}(A) = L \), where \( (L, \lambda) \) is a reticulation of \( A \) (which, as we know, is unique up to an isomorphism).

**Proposition 6.1.** Let \( A, B \) be residuated lattices, \( f: A \to B \) a morphism of residuated lattices and \( (\mathcal{L}(A), \lambda_A), (\mathcal{L}(B), \lambda_B) \) the reticulations of \( A \) and \( B \), respectively. Then there exists a unique bounded lattice morphism \( h: \mathcal{L}(A) \to \mathcal{L}(B) \) such that \( h \circ \lambda_A = \lambda_B \circ f \) (i.e. that makes the diagram below commutative).

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \lambda_A & & \downarrow \lambda_B \\
\mathcal{L}(A) & \xrightarrow{\mathcal{L}(f)} & \mathcal{L}(B) \\
\hline
h = \mathcal{L}(f)
\end{array}
\]
C. Mureşan

Proof. \((\mathcal{L}(B), \lambda_B)\) is a reticulation of \(B\), so it satisfies conditions (1)–(3). This and the fact that \(f\) is a morphism of residuated lattices (hence a bounded lattice morphism and a morphism of monoids) imply that the function \(\lambda_B \circ f\) satisfies conditions (1)–(3). \((\mathcal{L}(A), \lambda_A)\) is a reticulation of \(A\), so it satisfies (U) (see Proposition 4.1). The last two assertions imply that there exists a unique bounded lattice morphism \(h : \mathcal{L}(A) \to \mathcal{L}(B)\) such that \(h \circ \lambda_A = \lambda_B \circ f\).

Definition 6.1. With the notations in Proposition 6.1, set \(\mathcal{L}(f) = h\).

Notice that the definition above is in accordance with the definition of the reticulation functor that we gave in Section 2 (Definition 2.14).

Proposition 6.2. \(\mathcal{L}\) preserves injective morphisms.

Proof. Let \(A\) and \(B\) be residuated lattices, \((\mathcal{L}(A), \lambda_A), (\mathcal{L}(B), \lambda_B)\) their reticulations and \(f : A \to B\) an injective morphism of residuated lattices. We need to prove that \(\mathcal{L}(f)\) is injective.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \lambda_A & & \downarrow \lambda_B \\
\mathcal{L}(A) & \xrightarrow{h = \mathcal{L}(f)} & \mathcal{L}(B)
\end{array}
\]

Let \(l_1, l_2 \in \mathcal{L}(A)\) such that \(\mathcal{L}(f)(l_1) = \mathcal{L}(f)(l_2)\). Applying (4), we get that there exist \(a_1, a_2 \in A\) such that \(\lambda_A(a_1) = l_1\) and \(\lambda_A(a_2) = l_2\). Hence we have \(\mathcal{L}(f)(\lambda_A(a_1)) = \mathcal{L}(f)(\lambda_A(a_2))\), that is \(\lambda_B(f(a_1)) = \lambda_B(f(a_2))\). So \(\lambda_B(f(a_1)) \leq \lambda_B(f(a_2))\) and \(\lambda_B(f(a_2)) \leq \lambda_B(f(a_1))\), which, by applying (5), shows the existence of \(n, k \in \mathbb{N}^*\) such that \(f(a_1)^n \leq f(a_2)\) and \(f(a_2)^k \leq f(a_1)\). Since \(f\) is a morphism of residuated lattices, this amounts to \(f(a_1^n) \leq f(a_2)\) and \(f(a_2^k) \leq f(a_1)\). The injectivity of the lattice morphism \(f\) implies that \(a_1^n \leq a_2\) and \(a_2^k \leq a_1\). But this is equivalent (by (5)) with: \(\lambda_A(a_1) \leq \lambda_A(a_2)\) and \(\lambda_A(a_2) \leq \lambda_A(a_1)\), so \(\lambda_A(a_1) = \lambda_A(a_2)\), that is \(l_1 = l_2\).

Proposition 6.3. \(\mathcal{L}\) preserves surjective morphisms.

Proof. Let \(A\) and \(B\) be residuated lattices, \((\mathcal{L}(A), \lambda_A), (\mathcal{L}(B), \lambda_B)\) their reticulations and \(f : A \to B\) be a surjective morphism of residuated lattices. (See the diagram in Proposition 6.1). Then, since \(\lambda_B\) is surjective, \(\lambda_B \circ f\) is also surjective. But \(\mathcal{L}(f) \circ \lambda_A = \lambda_B \circ f\), so \(\mathcal{L}(f)\) is surjective.

In the examples below we shall use the second construction of the reticulation (see Section 2).
Proposition 6.4. \( \mathcal{L} \) does not reflect surjective morphisms.

Proof. Let \( A = \{0, a, b, c, d, 1\} \) be the residuated lattice with the partial order relation and operations presented in the Hasse diagram and tables below, respectively.

![Hasse diagram](image)

<table>
<thead>
<tr>
<th>( \rightarrow )</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>a</td>
<td>d</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>c</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>c</td>
<td>d</td>
<td>b</td>
<td>c</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>d</td>
<td>a</td>
<td>a</td>
<td>c</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>1</td>
</tr>
</tbody>
</table>

The principal filters of this residuated lattice are: \( \langle 0 \rangle = \langle b \rangle = A, \langle a \rangle = \langle c \rangle = \{a, c, 1\}, \langle d \rangle = \{d, 1\}, \langle 1 \rangle = \{1\} \), so \( \mathcal{L}(A) = \{(0), \langle a \rangle, \langle d \rangle, \langle 1 \rangle\} \), with the following lattice structure:

![Lattice structure](image)

The example above is example 5.3 from [15] (based on an example in [10]). Let \( B = \{0, a, b, c, d, e, f, g, 1\} \), with the following residuated lattice structure:

![Lattice structure](image)
Hence \( \langle 0 \rangle = \langle a \rangle = \langle c \rangle = \langle d \rangle = B, \langle b \rangle = \langle e \rangle = \{b, e, 1\}, \langle f \rangle = \langle g \rangle = \{f, g, 1\}, \langle 1 \rangle = \{1\} \). So \( \mathcal{L}(B) = \{\langle 0 \rangle, \langle b \rangle, \langle f \rangle, \langle 1 \rangle\} \), with the following lattice structure:

\[
\begin{array}{c|cccccc}
\langle 0 \rangle & \langle a \rangle & \langle c \rangle & \langle d \rangle & \langle 1 \rangle \\
\hline 
\langle 0 \rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & a & 0 & 0 & a & 0 & 0 & a \\
b & 0 & a & b & 0 & a & b & 0 & a \\
c & 0 & 0 & 0 & 0 & 0 & c & c & c \\
d & 0 & 0 & a & 0 & 0 & a & c & c & d \\
e & 0 & a & b & 0 & a & b & c & d & e \\
f & 0 & 0 & 0 & c & c & c & f & f & f \\
g & 0 & 0 & a & c & c & d & f & f & g \\
1 & 0 & a & b & c & d & e & f & g & 1 \\
\end{array}
\]

The second example above is also based on an example in [10].
A morphism from \( A \) to \( B \) that proves our proposition is, for instance, the function \( f \) given in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0</td>
<td>b</td>
<td>c</td>
<td>e</td>
<td>f</td>
<td>1</td>
</tr>
</tbody>
</table>

Here is the image of this morphism through the reticulation functor:

<table>
<thead>
<tr>
<th>( \langle x \rangle )</th>
<th>( \langle 0 \rangle )</th>
<th>( \langle a \rangle )</th>
<th>( \langle d \rangle )</th>
<th>( \langle 1 \rangle )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{L}(f)(\langle x \rangle) )</td>
<td>( \langle 0 \rangle )</td>
<td>( \langle b \rangle )</td>
<td>( \langle f \rangle )</td>
<td>( \langle 1 \rangle )</td>
</tr>
</tbody>
</table>
Notice that $L(f)$ is surjective; even bijective, actually. But $f$ is not surjective (as is obviously the case for any morphism from $A$ to $B$, due to the cardinalities of $A$ and $B$).

In order to find an example to prove this proposition, we have determined in fact all the morphisms from $A$ to $B$ and their images through the reticulation functor, by means of a small computer program.

**Proposition 6.5.** The reticulation functor is not full (see Definition 2.11).

**Proof.** We shall use example 5.1 in [15] (based on an example in [12]). Let $A = \{0, a, b, c, d, e, f, 1\}$, with the following residuated lattice structure:

\[
\begin{array}{cccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
a & d & 1 & a & a & f & f & f \\
b & e & 1 & 1 & a & f & f & f \\
c & f & 1 & 1 & 1 & f & f & f \\
d & a & 1 & 1 & 1 & 1 & 1 & 1 \\
e & b & 1 & a & a & a & 1 & 1 \\
f & c & 1 & a & a & a & a & 1 \\
1 & 0 & a & b & c & d & e & f
\end{array}
\]

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
a & 0 & c & c & c & 0 & d & d \\
b & 0 & c & c & c & 0 & d & d \\
c & 0 & c & c & c & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 & 0 & 0 & d \\
e & 0 & d & 0 & 0 & 0 & d & d \\
f & 0 & d & d & 0 & 0 & d & d \\
1 & 0 & a & b & c & d & e & f
\end{array}
\]

We have: \(\langle 0 \rangle = \langle d \rangle = \langle e \rangle = \langle f \rangle = A\), \(\langle a \rangle = \langle b \rangle = \langle c \rangle = \{c, b, a, 1\}\), \(\langle 1 \rangle = \{1\}\), hence $\mathcal{L}(A) = \{\langle 0 \rangle, \langle a \rangle, \langle 1 \rangle\}$, with the following lattice structure:

\[
\begin{array}{cccccccc}
< & 1 > \\
\hline
< & a > \\
< & 0 >
\end{array}
\]

By means of a small computer program, one can prove that there are only two morphisms of residuated lattices from $A$ to $A$: the identity (whose image through the reticulation functor is, of course, the identity of $\mathcal{L}(A)$) and the
following morphism: \( h : A \to A, h(0) = h(d) = h(e) = h(f ) = 0, h(a) = h(b) = h(c) = h(1) = 1 \) (whose image through the reticulation functor is: \( \mathcal{L}(f )((0)) = (0), \mathcal{L}(f )((a)) = \mathcal{L}(f )((1)) = (1) \)). But one can easily see that there are three bounded lattice morphisms from \( \mathcal{L}(A) \) to \( \mathcal{L}(A) \) (one for each possible value of such a morphism in \( (a) \)). Therefore the reticulation functor is not full.

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REFERENCES


