# Cramer-Rao Lower Bounds of Parameter Estimation for Frequency-Modulated Continuous Wave (CW) Lidar

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Signal processing is the key for frequency-modulated continuous-wave (FMCW) lidar, and the accuracy limits of parameters estimation are given by Cramer-Rao lower bound (CRLB). The aim for this work is to gain CRLB and evaluate the estimation algorithm by using CRLB. By introducing the generation of beat signal and the working principles of velocity and distance measurement, the joint probability density function (PDF) of sample parameter vector of intermediate frequency (IF) signal with Gaussian noise is established, then the CRLBs of velocity and distance estimation are obtained from the conversion of Fisher information matrix. It can be found from the theoretical analysis and simulation results that the CRLB can be effectively reduced by increasing the length of sampling data, decreasing the sampling frequency and improving the signal-to-noise ratio (SNR) of the system. A frequency offset correction (FOC) algorithm is discussed as well compared with CRLB. The difference between CRLB of velocity estimation and error of FOC algorithm is small and reduces from  $3.5 \times 10^{-5}$  to  $3.5 \times 10^{-7}$  m/s, as the SNR increases from -10.00 to 30.00 dB. The difference between CRLB of range estimation and error of FOC algorithm is large and constantly equals to 53.00 dB. It means FOC algorithm is suitable for velocity estimation of FMCW lidar, but there is a lot of room for improvement for range estimation. For having higher precise range estimation for FMCW lidar, the phase-based estimator is considered. The comparisons with CRLB show that the phase-based estimator could achieve high precise speed and range estimation. It is proven that the comparison of estimator and CRLB gives the basis for algorithm choice and improvement.

Keywords: Lidar, frequency-modulated continuous-wave (FMCW), Cramer-Rao lower bound, laser measurement, frequency offset correction algorithm, probability density function, analytical model, parameter estimation

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## **1 INTRODUCTION**

With the fast development of deep space exploration, modern intelligent transportation, high precision self-contained navigation and reverse engineering, higher accuracy real-time detection of velocity and distance is required. Frequency-modulated continuous-wave (FMCW) lidar is one of the most precise instruments for velocity and distance measurement, which operates by using linear FMCW laser as the carrier. Comparing with the traditional FMCW radar that works in the microwave frequency band, it has the advantages such as imaging, higher resolution of velocity, distance and angle, and much wider modulation bandwidth [1, 2]. Due to the advantages it has broad application prospects in the fields like automatic landing, rendezvous and docking for aircraft, high precision liquid level measurement, long-endurance high precision navigation, three-dimensional (3-D) scanning for large size artefacts, and so on [3-6].

Because of the influence of external environment and internal factors, it is impossible to get the pure signal without noise in the practical applications. Signal processing method used to extract the useful parameters is the key for FMCW lidar, and research on it has become a hot spot. Though many algorithms have been proposed, there is lack of method to evaluate these algorithms and to tell the room for improvement. Cramer-Rao lower bound (CRLB), the most effective unbiased variance of estimation algorithm, is usually used as the precision limit. In 1975, Rife [7] receives the CRLB of the parameters estimation of a single tone from a finite number of noisy discrete-time observations. In 1996, Besson and Boorstyn [8] obtained the closed-formed expressions of CRLB on the lidar accuracy of amplify and Doppler shift. In 2008 Wang *et al.* [9] gave the CRLB of parameter estimation when there exists large acceleration. Considering the influence of acceleration and the Gaussian envelope of laser Doppler velocimeter, the CRLB is analysed in 2010 by Zhou *et al.* [10].

In this work the closed-formed expressions of CRLB of FMCW lidar are obtained for the first time. According to the beat signal model with Gaussian noise and joint probability density function of the parameter vector, Fisher information matrix is built. After transformation, the CRLBs of velocity and distance are obtained. Numerical simulations are done to compare the error of frequency offset correction (FOC) algorithm and phase-based algorithm with the CRLB.

### 2 THE CRAMER-RAO LOWER BOUND (CRLB)

Due to the advantages of high sensitivity, coherent detection is thought to be one of the most effective techniques for optical spectrum detection, and is usually used as a solution for FMCW lidar to detect the echo signal. After equal interval sampling, the beat signal,  $z_n$ , can be written as

$$z_n = A \exp\left\{j\left[\frac{2\pi f_d\left(nt_d\right) + 4\pi B_f nt_d\tau}{N + 2\pi f_l\tau + \varphi_o}\right]\right\} + w_n = x_n + jy_n \tag{1}$$

where n = 0, 1, 2, ..., N - 1; *A* is the amplitude of the beat signal;  $f_d$  is the Doppler shift caused by the movement of target;  $t_d$  is the sampling interval;  $B_f$  is the bandwidth of modulation;  $\tau$  is the delay of echo signal; *N* is the number of samples;  $f_l$  is the starting frequency of laser source;  $\varphi_o$  is the initial phase of beat signal;  $w_n$  is additive plural white Gaussian noise with variance of  $\sigma_w^2$ ;  $x_n$  and  $y_n$  are the real part and imaginary part of  $z_n$ , respectively. Because the sample period is very short and the manoeuvrable process usually does not last long, we are assuming that all the acceleration related items have been compensated for [11] and the Doppler shift function,  $f_d(nt_d)$ , is constant.

The signal-to-noise ratio (SNR) can be calculated by

$$SNR = \frac{A^2}{\sigma_w^2} \tag{2}$$

There are four unknown parameters in Equation (1), and the parameter vector,  $\Gamma$ , is given by

$$\Gamma = \left[ f_d, \ \tau, \ A, \ \phi_0 \right]^T \tag{3}$$

The joint probability density function (PDF),  $f(\overline{z_n}; \overline{\Gamma})$ , of the observations is obtained:

$$f\left(\overline{z_{n}};\overline{\Gamma}\right) = \left(\frac{1}{\pi\sigma_{w}^{2}}\right) \exp\left(-\frac{1}{\sigma_{w}^{2}}\sum_{n=0}^{N-1} \left\{ \begin{bmatrix} x_{n} - A\cos\left(\frac{2\pi f_{d}\left(nt_{d}\right)}{N+2\pi f_{l}\tau+\varphi_{o}}\right) \\ y_{n} - A\cos\left(\frac{2\pi f_{d}\left(nt_{d}\right)}{N+2\pi f_{l}\tau+\varphi_{o}}\right) \end{bmatrix}^{2} + \right\} \right\}$$

$$\left[ y_{n} - A\sin\left(\frac{2\pi f_{d}\left(nt_{d}\right)}{N+2\pi f_{l}\tau+\varphi_{o}}\right) \right]^{2} + \right]$$

$$\left( (4)$$

According to the theory of statistics, aiming to achieve the CRLB of a FMCW lidar requires establishment of Fisher information matrix first. Fisher information matrix is a symmetric matrix and each element in the matrix can be calculated by

$$J_{ij} = \frac{\partial^2 \ln\left[f\left(\overline{z_n};\Gamma\right)\right]}{\partial\Gamma_i \partial\Gamma_j} \tag{5}$$

So now the elements of the Fisher information matrix are

$$J_{11} = -\frac{8\pi^2 t_d^2 A}{\sigma_w^2} \sum_{n=0}^{N-1} n^2 \left( x_n \cos \Omega + y_n \sin \Omega \right)$$
(6a)

$$J_{22} = -\frac{2A}{\sigma_w^2} \sum_{n=0}^{N-1} \left( 4\pi B_f n t_d / T_m + 2\pi f_l \right)^2 \left( x_n \cos \Omega + y_n \sin \Omega \right)$$
(6b)

$$J_{33} = -\frac{N(N-1)}{\sigma_w^2} \tag{6c}$$

$$J_{44} = -\frac{2A}{\sigma_w^2} \sum_{n=0}^{N-1} \left( x_n \cos \Omega + y_n \sin \Omega \right)$$
(6d)

$$J_{12} = J_{21} = -\frac{4\pi t_d A}{\sigma_w^2} \sum_{n=0}^{N-1} n \left( 4\pi B_f n t_d / T_m + 2\pi f_l \right) \left( x_n \cos \Omega + y_n \sin \Omega \right)$$
(6e)

$$J_{13} = J_{31} = -\frac{4\pi t_d}{\sigma_w^2} \sum_{n=0}^{N-1} n \left( x_n \sin \Omega - y_n \cos \Omega \right)$$
(6f)

$$J_{24} = J_{42} = -\frac{2A}{\sigma_w^2} \sum_{n=0}^{N-1} (x_n \cos\Omega + y_n \sin\Omega) (4\pi B_f n t_d / T_m + 2\pi f_l)$$
(6g)

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$$J_{23} = J_{32} = -\frac{2}{\sigma_w^2} \sum_{n=0}^{N-1} (x_n \sin \Omega - y_n \cos \Omega) (4\pi B_f n t_d / T_m + 2\pi f_l)$$
(6h)

$$J_{34} = J_{43} = -\frac{2}{\sigma_w^2} \sum_{n=0}^{N-1} \left( x_n \sin \Omega - y_n \cos \Omega \right)$$
(6i)

where  $\Omega = \frac{2\pi f_d (nt_d) + 4\pi B_f nt_d \tau}{N + 2\pi f_f \tau + \varphi_o}$ .

From Equation (1) we know that  $x_n = A\cos\Omega$  and  $y_n = A\sin\Omega$ . In addition, it can be found that  $N \Box 1$  and  $f_l \Box B_f nt_d / T_m$ , so now the matrix can be simplified to

$$J = \begin{bmatrix} -\frac{8\pi^{2}t_{d}^{2}A^{2}N^{3}}{3\sigma_{w}^{2}} & -\frac{4A^{2}f_{l}N^{2}\pi^{2}t_{d}}{\sigma_{w}^{2}} & 0 & -\frac{2A^{2}N^{2}\pi t_{d}}{\sigma_{w}^{2}} \\ -\frac{4A^{2}f_{l}N^{2}\pi^{2}t_{d}}{\sigma_{w}^{2}} & \frac{8A^{2}f_{l}^{2}N\pi^{2}}{\sigma_{w}^{2}} & 0 & -\frac{4A^{2}f_{l}N\pi^{2}}{\sigma_{w}^{2}} \\ 0 & 0 & -\frac{N^{2}}{\sigma_{w}^{2}} & 0 \\ -\frac{2A^{2}N^{2}\pi t_{d}}{\sigma_{w}^{2}} & -\frac{4A^{2}f_{l}N\pi}{\sigma_{w}^{2}} & 0 & -\frac{2A^{2}N}{\sigma_{w}^{2}} \end{bmatrix}$$
(7)

According to the Cramer-Rao inequality, the CRLB should satisfy the following condition [12]:

$$\sigma_{CRB|\Gamma_i}^2 = \left(\Gamma_i\right)_{CRLB} \ge \left|\left(J^{-1}\right)_{ii}\right| \tag{8}$$

where  $(J^{-1})_{ii}$  stands for the *i*<sup>th</sup> diagonal element of inverse matrix of *J*.

After the transformation the CRLBs of the Doppler shift and the delay of echo signal can be obtained:

$$\sigma_{CRB|f_d} = \frac{1}{\pi t_d N} \sqrt{\frac{3}{2N \cdot SNR}}$$
(9)

and

$$\sigma_{CRB|\tau} = \frac{1}{4f_{l}\pi\sqrt{N\cdot SNR}} \tag{10}$$

According to the conditions of the FMCW lidar, the velocity estimation, v, is  $v = \lambda f_d / 2 \cos \theta$  and the distance estimation, *R*, is  $R = \tau c / 2$ , where  $\lambda$  is the wavelength and  $\theta$  is the beam pointing angle. So, the CRLBs of the velocity and the distance estimation can be gained:

$$\sigma_{CRB|\upsilon} = \frac{\lambda}{2\pi t_d N \cdot \cos\theta} \sqrt{\frac{3}{2N \cdot SNR}}$$
(11)

and

$$\sigma_{CRB|R} = \frac{\lambda}{8\pi\sqrt{N \cdot SNR}}$$
(12)

There are several parameters have impact on the CRLBs. The values of  $\lambda$ and  $\theta$  are determined by the hardware of the FMCW lidar, while sampling rate, sample number and SNR are variables that can be set in the algorithm. Simulations are made to illustrate the dependence of CRLB on the sampling rate, sample number and SNR. The results are shown in Figure 1. In Figure 1(a) the CRLB of velocity estimation at the sample number of 256, 512, 1024 and 2048 are illustrated. The curves show the CRLB of velocity estimation decreases as sample number and SNR increase. Figure 1(b) shows the CRLB of velocity estimation at the sampling rate of 5, 10, 20 and 100 MHz. It tells the CRLB of velocity estimation decreases as sampling rate reduces. To have small CRLB of velocity estimation, we should have small sampling rate, large sample number and SNR. From Figure 1(c) we can see that curves of CRLB of distance estimation decrease as sample number and SNR increase. Aiming to achieve higher precision requires smaller CRLB of range estimation, it is better to choose sample number and the SNR as large as possible.

## **3 SIMULATION PROCEDURE AND RESULTS**

Frequency extraction from the noisy signal is common in use and many algorithms based on fast Fourier transform (FFT) have been proposed. Take the frequency estimation algorithm based on frequency offset correction (FOC) as an example. The calculation flowchart of FOC algorithm is shown in Figure 2.

Since the autocorrelation could greatly reduce the Gaussian white noise, it is an effective method to improve the SNR. Due to the limit of the length of sampling data, there is residual noise left. The standard deviation of frequency estimation is given by



#### FIGURE 1

Graphs showing (a) CRLB of velocity estimation varying with SNR at different sample number, (b) CRLB of velocity estimation varying with SNR at different sampling rate and (c) CRLB of range estimation varying with SNR at different sample number.



FIGURE 2 Calculation flowchart of FOC algorithm.

$$\sigma_{FOC|f} = \frac{1}{2\pi t_d \left(N - m\right) \sqrt{m \cdot SNR}}$$
(13)

where *m* represents the delay time. From Equation (10) we can see the standard deviation of frequency estimation associates with the delay time. In the algorithm, *m* has impact on  $\sigma_{FOC|f}$ , and it should be optimized.  $\frac{d\sigma_{FOC|f}}{dm} = 0$ is established and the optimized *m* equals to *N*/3. Now Equation (13) can be written as

$$\sigma_{FOC|f} = \frac{3}{4\pi t_d N} \sqrt{\frac{3}{N \cdot SNR}}$$
(14)

Then standard deviations of the velocity and distance estimator based on PDM are

$$\sigma_{FOC|v} = \frac{3\sqrt{3\lambda}}{8\pi t_d N \sqrt{N \cdot SNR} \cdot \cos\theta}$$
(15)

and

$$\sigma_{FOC|R} = \frac{ct_d N}{B_f} \sigma_{FOC|f} = \frac{3\sqrt{3}c}{4\pi B_f \sqrt{N \cdot SNR}}$$
(16)

where *c* is the speed of light.

Simulations are made with different SNR to compare the CRLB and the error of the FOC algorithm, and the main parameters are seen in Table 1. The results are shown in Figure 3. From Figure 3(a), it can be seen that the difference between CRLB of velocity estimation and error of FOC algorithm is small. As the SNR increases from -10.00 to 30.00 dB, the error of FOC

TABLE 1 Main parameters used.

Parameter (Unit)	Value
Number of samples	1024
Sampling rate (MHz)	20
Beam pointing angle (°)	0
Bandwidth (GHz)	10
Wavelength (nm)	1550



FIGURE 3

Graphs showing (a) velocity error of FOC algorithm and CRLB of velocity estimation varying with SNR and (b) range error of FOC algorithm and CRLB varying with SNR.

algorithm is approaching CRLB and the difference reduces from  $3.5 \times 10^{-5}$  to  $3.5 \times 10^{-7}$  m/s, which is small enough to be ignored in most cases. So, the FOC algorithm could achieve velocity measurement with small error, and it is a high precision velocity estimator for FMCW lidar. In Figure 3(b), the difference between CRLB of range estimation and error of FOC algorithm is large, and equals to 53.00 dB constantly no matter what value the SNR has. Though it achieves the measurement with millimetre scale, it is not a high precision range estimator.

According to the analysis, if higher precise range estimation is required then another algorithm should be considered. A thought to solve the problem is application of phase-based estimator. Its velocity and range estimation standard deviation can be expressed as [13]

$$\sigma_{P|v} = \frac{\lambda}{\sqrt{2\pi t_d N \sqrt{N \cdot SNR} \cdot \cos \theta \cdot \sin c \left(\gamma\right)}}$$
(17)

and

$$\sigma_{P|R} = \frac{\lambda}{8\pi\sqrt{N \cdot SNR} \cdot \sin c\left(\gamma\right)}$$
(18)

where  $\gamma$  is the difference between the ideal maximum frequency and the maximum spectral line of the discrete Fourier transform (DFT), and it is in the range -0.5 to 0.5.

Comparison of the CRLB and the error of phased-based algorithm are done and the results are shown in Figure 4. It can be seen from Figure 4(a) the difference between CRLB of velocity estimation and error of phase-based algorithm is small and increases as  $|\gamma|$  arises. As it obeys the uniform distribution in its scope, the difference is smaller than the value when  $|\gamma| = 0.5$ . Comparing with the FOC algorithm, it may have some loss on velocity measurement precision, but it has much higher precision on range estimation. In Figure 4(b), we can see the difference between CRLB of range estimation and error of phase-based estimation is greatly reduced. It also increases as  $|\gamma|$  arises, and is smaller than 1.96 dB when  $|\gamma| = 0.5$ . So phase-based algorithm could achieve velocity and range measurement with small error, and it is a high precision velocity and range estimator for FMCW lidar.

## **4 CONCLUSIONS**

In summary, the closed-formed expressions Cramer-Rao lower bound (CRLB) of velocity and range estimation for frequency-modulation continuous-wave



FIGURE 4

Graphs showing (a) velocity error of phase-based algorithm at different  $\gamma$  and CRLB of velocity estimation varying with SNR and (b) range error of phase-based algorithm and CRLB varying with SNR.

(FMCW) lidar are obtained. Aiming to achieve higher precision, it is better to have small sampling rate and large sample number and signal-to-noise ratio (SNR). The comparisons of CRLB and error of FOC algorithm are done by numerical simulations. The conclusions can be drawn that frequency offset correction (FOC) algorithm is a high precision velocity estimator for FMCW lidar, but not a range estimator with high precision. Error of phase-based estimator is analysed for higher precise range estimation, and comparisons with CRLB are made. The results reveal that phase-based algorithm is a high precise velocity and range estimator for FMCW lidar.

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# NOMENCLATURE

- A Amplitude of the beat signal (dB)
- $B_f$  Bandwidth of modulation (Hz)
- c Speed of light (m/s)
- $f_d$  Doppler shift (Hz)
- $f_l$  Starting frequency of the laser source (Hz)
- *m* Delay time (seconds)
- *N* Number of samples
- *R* Distance estimation (m)
- $t_d$  Sampling interval (seconds)
- $w_n$  Additive plural white Gaussian
- $x_n$  Real part  $z_n$
- $y_n$  Imaginary part of  $z_n$
- $z_n$  Beat signal (Hz)

# **Greek symbols**

- $\gamma$  Difference between the ideal maximum frequency and the maximum spectral line of the discrete Fourier transform (DFT)
- Γ Parameter vector
- $\phi_o$  Initial phase of the beat signal (Hz)
- θ Beam pointing angle (°)
- $\lambda$  Wavelength (m)
- $\tau$  Delay of the echo signal (seconds)

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