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Thermal and Superthermal Income Classes in a Wealth Alike Distribution Generated by Conway's Game of Life Cellular Automaton

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By means of a data analysis, we study and confirm that the complexity of Conway's Game of Life cellular automaton (GoL) is enough to reproduce the statistical universal properties of wealth distributions as they are observed in real economic data. In GoL's Economy, we interpret each rebirth of a cell as the event of an agent gaining a "monetary unit". We show that the GoL's "wealth distribution" generated in this way is compatible with the exponential/gamma distribution observed in real economical complex systems for the low and medium classes of the population of a country or a society (thermal class). Furthermore, GoL also reproduces the power law asymptotic behavior of real wealth distribution corresponding to the richest sector of the population (superthermal class). Analyses of Gini index and of the stationarity of GoL generated wealth distribution were also performed.

Keywords: Game of life, econophysics, wealth distribution, gamma distribution, power law distribution.

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1 INTRODUCTION

1.1 Cellular Automata

In their most basic form, cellular automata are discrete deterministic dynamical systems in space and time whose evolution is defined in terms of local interactions. Cellular automata were introduced in the late forties by John von Neumann and Stanislaw Ulam as a tool to understand the biological mechanisms of self-reproduction [21, 36]. Due to its intrinsically simple mathematical structure and their success to model emergent phenomena in many kind of complex systems (physical, chemical, economical and biological, architecture design in parallel computing, traffic models, programming media, etc. [37]), cellular automata are an important research area belonging simultaneously to Mathematics and the Complexity Sciences.

In the seventies and thanks to an article written by Martin Gardner [25] on a very simple but of high complex behaviour cellular automaton called the “Game of Life”, proposed by J. H. Conway in the late sixties, cellular automata reached with high success the community of mathematics and computing fans. In fact a Game of Life generated pattern called “the glider” is the emblem of the hacker culture. For a great collection of papers covering different aspects of the “Game of Life” cellular automaton, see [1].

The Game of Life

“The Game of Life” cellular automaton, from now on denoted “GoL”, consists of a two-dimensional rectangular lattice with $N \times N$ cells. Each cell may adopt one of two possible states: one (*alive or white*) or zero (*dead or black*). In the most simple case as the one analyzed in the present work, the Moore neighborhood with unit radius it is formed by a central rectangular cell and its eight nearest neighbors. The Moore neighborhood is used to mediate interactions between cells at time t and to obtain the next state of its central cell at time step $t + 1$. GoL evolves by applying the following set of rules to every one of the $N \times N$ cells of the lattice:

- Births: a cell that is dead at time t will come back to life at time $t + 1$, if exactly three of its eight neighbors are alive at time t.
- Survival: a cell will survive from time t to time $t + 1$, if two or three of its neighbors are alive at time t.
- Death: a cell can die by:
  - Overpopulation: if a cell is alive at time t and more than three of its neighbors are also alive, the cell will die at time $t + 1$.
  - Exposure: if a live cell at time t has less than two living neighbors, it will die at time $t + 1$.

1.2 Complexity, Economy and Wealth Distribution

The strong resemblance between the behavior of certain economic phenomena and the physics problem of analyze systems with an extremely big number of interacting particles, has brought together the joint interest and cooperation of specialists in areas such as mathematics (information theory and stochastic processes), statistical physics and computer science, achieving with this collaboration a deeper capacity of analyze and to study financial and economic systems. Complex systems theory has been the tool for the study of several related economic problems, such as the behavior of price fluctuations in stock markets, network analysis of economic systems, financial crises and the distribution of wealth in a society or country [16, 22, 23, 29].

Appearance of distributions with power laws is ubiquitous in the aforementioned economic phenomena and it is an important characteristic of complex systems and critical phenomena. Underlying mechanisms that produce power laws distributions are also of great interest for physicists and scientists interested in Complex Systems. The most well known mechanism of power law emergence presents in critical phenomena, such as phase transitions, percolation and self organized criticality (SOC), although many more mechanisms to generate power law distributions have been discovered [10, 20, 26, 40].

In economics, besides the appearance of power law distributions, a wide collection of universal, empirical statistical properties emerges, and they are collectively known by the name of “stylized facts”, which are empirical findings that are always present in all economic and financial systems. They include, but are not limited to, absence of correlations in price variations, long-range correlations of prices absolute values, volatility grouping, Gaussian aggregation, etc. [28, 33].

On the other hand, from the microscopic point of view, in the area of financial market models, analysis techniques called microscopic simulations (MS) or agents based models (ABM) have been developed [14, 18] to construct artificial markets, which consist of a big number of agents, with preset interactions and used to investigate the global evolution of the real system. This line of research has generated several models capable of reproducing the stylized facts observed in real economic phenomena [13, 38].

The fact that a relatively simple, microscopic simulation can reproduce the stylized facts observed in the real economy, with all its complexity, is promising and has led to the task of building artificial stock markets through agents models and cellular automata models as GoL and others [17, 24, 34, 39].

In the particular case of the problem of understanding the distribution of wealth in a country or society, by simulating closed economical systems where number of agents and money are conserved, and considering the money as the analog of energy, the emerging equilibrium probability distribution of wealth is the exponential Boltzmann-Gibbs distribution [5], in agreement with classical statistical mechanics ensemble theory [12].

Now, it is a well known empirical fact that a combination of two distributions describes the individual distribution of wealth of a country or society: the wealth distribution of the lower and medium income sectors of the population follows a gamma distribution and wealth distribution of the highest income sector of the population decays as a power law [6]. For a complete discussion of this issue with an intensive bibliography, see [4, 23].

The joint emergence of these particular two distributions in a single system it is not a very common phenomenon. Regarding to the socio-economic problem of wealth distribution, in analogy with [7], and in a very interesting paper [27], V. Yakovenko and Sylva refer to the exponential and Pareto components of wealth distribution as thermal and superthermal income classes respectively.

Reference [23] presents and discusses different multi-agent exchange models, where one in particular, that in presence of savings, reproduces the real wealth distribution property of having the low and medium income sectors well described by a gamma distribution and still the wealth of the richest part of the populations decaying as a power law. Same last cited reference presents the pertinent bibliography in the topic of gas like and kinetic exchange models.

It is pertinent here again to remark that besides wealth distribution, the simultaneous emergence of the combination of exponential/gamma and power law distributions in a single system or phenomenon is not very common, we can point out other well known examples of the coordinated appearance of these two distributions: firstly observed in plasma physics and astrophysics phenomena [7, 19, 32], in heavy ion physics [2], songs rankings [8], the Ising Model [34] and in GoL generated wealth distribution as is shown in the present paper.

We show in this research that the wealth distribution generated by GoL also displays a thermal and a superthermal income classes.

This paper is the natural continuation of [9] and [34], where we demonstrate that GoL fluctuations studied as a diffusion process behave as a geometrical Random Walk and reproduces the stylized facts of financial time series respectively.

2 CONSTRUCTION OF THE DATA SAMPLE AND THE GOL GENERATED "WEALTH DISTRIBUTION"

The data sets analyzed in this study were generated from the dynamics of Conway's Game of Life. Our implementation of the Game of Life uses periodic boundary conditions, an initial state with a certain percentage of living cells randomly selected and uniformly distributed in a lattice size of $N \times N = 3000 \times 3000$ cells. The system is let to evolve, and at each time step, states of all cells in the lattice are updated according to GoL's rule. For each cell, we keep a counter that is increased by one unit every time a dead cell becomes alive, and this increment is interpreted as the cell's gaining of an unit of money.

After 500 time steps, we obtain a data set with $N \times N = 3000 \times 3000 = 9000000$ entries, with records of the number of times each cell was brought back to life, i.e. the "wealth" of every cell. This calculation is made with initial densities of randomly allocated living cells of 20%, 30%, 40% and 50%, in order to obtain four data sets to be analyzed. Finally, and in order to obtain the wealth distribution with a minimum wealth equal to zero, we locate the cell with minimum "richness" and subtract this wealth value to every wealth value of all other cells in the lattice.

As an illustration of the above explained, panels of Figure 1 display the histogram of the wealth distribution, where wealth is denoted by w, obtained from GoL dynamics with a 3000×3000 lattice size and 20% initial alive cells. Upper panel subfigures, 1(a) to 1(c), show the overall of the corresponding wealth distribution in linear, log-vertical and log-log scales respectively; whereas lower panel subfigures 1(d) to 1(f) show the same distribution for the cases where $w < 60$ in linear and log-vertical scales and its plot in the region $w > 60$ in a log-log scale respectively. From the last two subfigures 1(e) and 1(f), it is evident that the wealth distribution generated by GoL contains an exponential and an asymptotic power law parts. The red line is not a fit and is only used to evidence the exponential and power law decays. In subsection 3.2, we will show the corresponding fits of the gamma and log-normal distributions for the poorer and medium regions of wealth distribution and the fit of the power law to the richest segment of population. Fits were performed using the CERN numerical minimization computer program Minuit, embeded as a C++ class in the ROOT software package [31]. We do not perform the exponential fit in our analysis because gamma distribution encompasses as a factor this model.

2.1 Decreasing noise induced by periodical configurations: Limiting the number of generations

We have chosen by trial and error to stop GoL simulation after 500 generations. This is done because selecting a bigger number of time steps implies recording a higher number of events in the tail of the wealth distribution coming from persistent configuration or patterns of cells that change state periodically in time, as for example the "Blinker", which we consider for the goals of this paper, as a source of noise, since cells in these patterns extend the tail of the wealth distribution to infinity when the number of times steps

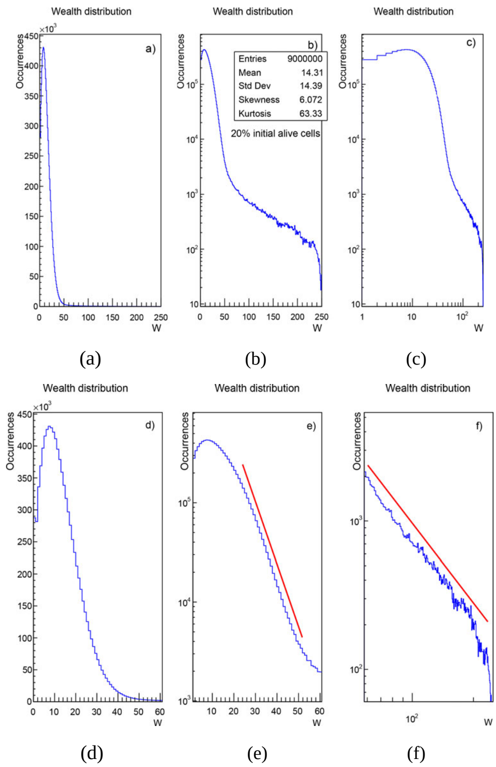


FIGURE 1
Wealth distribution generated with a lattice size of nine millions of cells and 500 time steps. Different scales and regions are shown. (a) to (c), overall of the wealth distribution in scales linear, log-vertical and log-log respectively. (d) and (e) linear and log-vertical scales for the region $w < 60$. (f) Wealth distribution for region $w > 60$ in a log-log scale. Red straight lines are not fits, they are only useful to evidence the exponential and power law decays. Corresponding fits are shown in subsection 3.2.

increases without limit. Other reason to select a short number of time steps to generate the data samples, is because after some time, GoL reaches a quiescent state, far away of any possible critical state, the one of interest, with power law signatures [30].

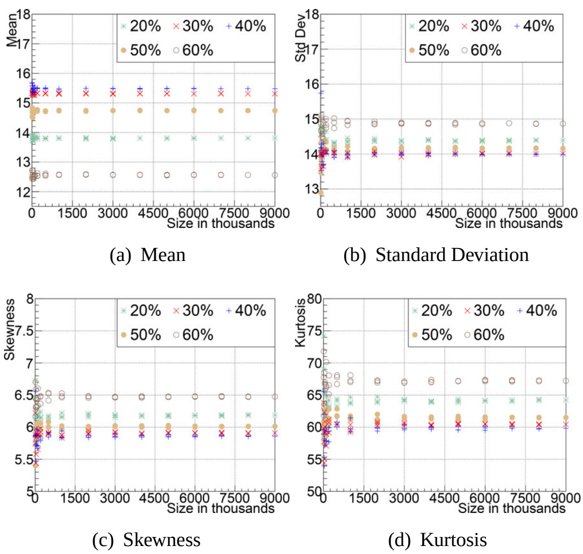
3 DATA ANALYSIS

The data analyses performed to the generated data, consist first of a stationarity distribution analysis. This study assures our results do not depend of the lattice selected size. The second analysis consists in showing that the wealth distribution generated by GoL is consistent with the empirical distribution observed in different complex systems, in particular with the shape of the wealth distribution observed in real data of socio-economic systems. Finally, a calculation of the Gini index of the GoL-generated wealth distribution is presented, useful to compare "inequality" of GoL-generated wealth distribution with real data.

3.1 Spatial stationarity of the empirical wealth distribution

In this subsection we show that 500 time steps, GoL generated random subsamples with different sizes, obtained from the 3000×3000 elements data samples, for a fixed initial density of alive cells, maintain their distribution unchanged. We select, for each one of our samples with initial densities of alive cells, 20%, 30%, 40% and 50% fifteen random subsamples with the following sizes: 10×10^3, 30×10^3, 50×10^3, 100×10^3, 200×10^3, 500×10^3, 1×10^6, 2×10^6, 3×10^6, 4×10^6, 5×10^6, 6×10^6, 7×10^6, 8×10^6 and 9×10^6 cells. The last number corresponds to the overall of the generated data of a particular GoL generated data sample.

In order to assure the spatial stationarity of our data, we plotted size, i.e. the number of cells of the the previous specified subsamples vs the four first central moments of these data sets, i.e. their mean, standard deviation, kurtosis and skewness. The experiment was repeated three times. Corresponding plots are displayed in Figure 2. We can see the values of the four central



(a) Mean (b) Standard Deviation

(c) Skewness (d) Kurtosis

FIGURE 2
Four first statistical central moments dependency on size of 15 the different generated data subsamples for the four initial densities of alive cells. The experiment was repeated three times.

moments become virtually constant as sample size increases, depending only of the initial density of alive cells.

We are aware that by following this empirical methodology to assure the stationarity of a distribution, would imply to show all distribution central moments, i.e., an infinite number of them, are constant, clearly an impractical task. However, for the most of practical applications it is enough to work with the first four central moments of a distribution.

3.2 Fitting the GoL generated wealth distribution

In this subsection, we show that GoL-generated wealth distribution has the same shape that the one universally observed in real socio-economic systems. To achieve this goal, we analyze as described below, the sets of four GoL generated wealth distributions with 20%, 30%, 40% and 50% of initial randomly allocated living cells, all with lattice sizes of 3000×3000.

The most well known statistical models used by social scientists and econophysicists to fit wealth distribution are the following:

M.1 The Gibbs-Boltzmann distribution (exponential):

$$f(x) = K_{GB} e^{-x/T}$$

with decay constant $\frac{1}{T}$.

M.2 Gamma distribution:

$$f(x) = K_G x^{k-1} e^{-x/\theta}$$

where k and θ are the gamma shape and form parameters respectively.

M.3 The log-normal distribution:

$$f(x) = K_{LN} \frac{1}{x} e^{-(\ln x - m)^2/2n^2}$$

where parameters m and n are the mean and deviation of the distribution. And finally,

M.4 The power law distribution:

$$f(x) = K_{PL} x^{-\alpha}$$

where α is called the power value or shape parameter of the distribution.

Here K_{GB}, K_G, K_{LN} and K_{PL} are treated as constants and all parameters and these constants are obtained from the fitting procedure described next.

Models M.1 to M.3 describe the wealth distribution of the low and medium classes segments of the population. Is well known that the exponential or Gibbs-Boltzmann model M.1, as statistical physics postulates describes the energy distribution in a system with many particles where number of particles and energy are conserved, i.e. the system is closed and the change of energy of individual particles is only due to particle to particle interactions (exchange) and reallocation of energy or, in our case wealth. Gamma model M.2 is the one econophysicists prefer. This model is favored for physicists because includes the exponential factor emerging from exchange and reallocation of wealth or energy from statistical physics and a second power law factor statistically describing other phenomena, such as savings or enrichment (wealth condensation). Empirically, the gamma distribution seems to describe correctly the wealth of low and medium income segments of a population [3, 4, 42]; Log-Normal model M.3 is preferred by economists and social scientist to statistically describe wealth of a population [11, 43], and finally Model M.4 describes correctly the wealth distribution of the richest sector of a population, It was used firstly by Vilfredo Pareto with the purpose of describing wealth economic data of diverse societies [41].

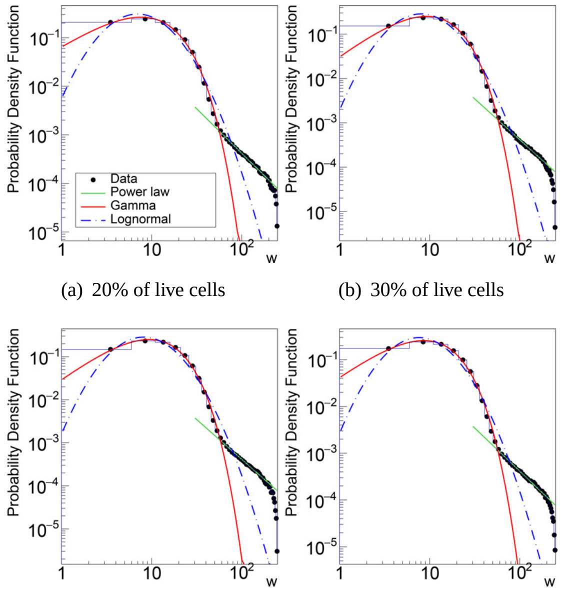
In this paper we do not present a fit of the Gibbs-Boltzmann model M.1 to the GoL-generated wealth distribution due to the following two reasons:

1. M.1 model describes wealth distribution of very simple agents models, i.e. ideal-gas alike models and does not describe correctly the concave shape of GoL-generated wealth distribution observed in figure 3, meaning that the lowest income segment of population is not well described by this model.
2. Gamma model M.2 encompasses the exponential factor corresponding to model M.1 (contains it as a factor).

Fits of models M.3 to M.4 to the four data sets of GoL-generated wealth distributions are shown in Figure 3. In all subfigures we can clearly observe an asymptotic power law distribution, which corresponds to the "richest" segment of the GoL-generated wealth distribution and apparently good Gamma and log-normal fits to the sector of medium income and poorest agents. The Boltzmann-Gibbs distribution was not performed for the reasons explained above.

Methodology and quality of fits

In this subsection we assess the quality of the fits shown in the figure 3 separately, i.e. the quality of statistical models fitting empirical data for the thermal (log-normal/gamma) and superthermal (power law) classes.



(a) 20% of live cells
(b) 30% of live cells
(c) 40% of live cells
(d) 50% of live cells

FIGURE 3
Fits of wealth distribution for our data generated sets. An asymptotic power laws fits well the richest regions of population (right of the graphs). Log-Normal and Gamma fits were applied to the lower and medium income region. All fits are displayed separately in figures 5 and 6 below.

Since each data sample has 9×10^9 entries, a really huge statistics, small deviation of the theoretical fitted model from the empirical distribution are important and it is very hard to perform a direct and standard goodness of fit test and/or the obtained result may be misleading [35].

Table 1 shows the analysed statistics for low/medium income (thermal) and higher income (superthermal) regions. The cut off value separating these

% initial alive cells	**Low/medium income ($w < 60$)**	**High income ($w > 60$)**	**Power law fitted region ($60 < w < 200$)**
20%	8908312	91688	83456
30%	8909102	90898	83861
40%	8909463	90537	83421
50%	8908727	91273	83911

TABLE 1
Statistics (number of records) of analyzed samples for a cut off value of $w = 60$ monetary units, separating thermal (low/medium) and superthermal (high) income classes. Total number of entries for each sample is 9×10^9 records. Last column shows the region of the superthermal income class where the power law model was fitted to data.

classes is $w = 60$ and was selected by eye. In fact the separation between these regions is very clear and no further statistic procedure to estimate the cut off value was needed [15]. An additional cut-off in the high income region was applied, limiting our analysis of this superthermal income class to the region $60 < w < 200$. This is useful to eliminate finite lattice, border size effects and high periodicity, non complex events as for example "blinkers" and other periodic patterns, mentioned in subsection 2.1

Fitting GoL-generated Wealth distribution: Thermal income class

In this case, and especially due to the high statistics of the low/medium income region, it is better to work on the Cumulative Distribution Function, defined for a continuous random variable W, in our case cells individual wealth distribution, in the usual way:

$$WCDF(w) := Prob\{W < w\} = \int_0^w f(x)dx \tag{1}$$

Where $w \geq 0$, and $f(w)$ is the wealth Probability Density Function (PDF) obtained from GoL generated data and plotted for each one of our four data samples in figure 3. Wealth Cumulative Distribution Function (CDF) denoted WCDF for the four data samples analysed are displayed in figure 4

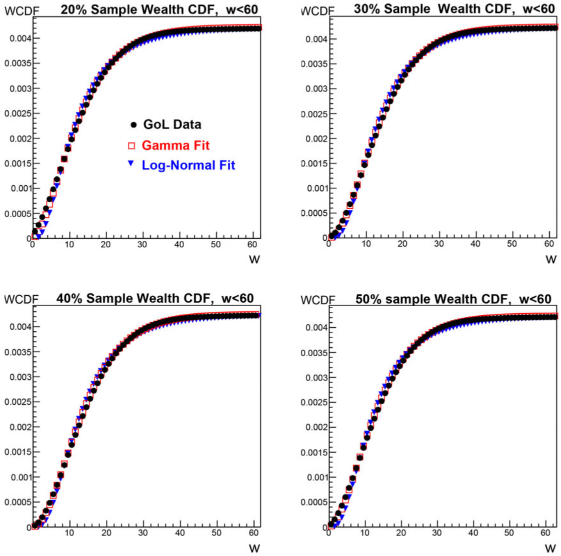


FIGURE 4
Wealth CDF, denoted WCDF for GoL generated data and Gamma and Log-Normal Fits for 20%, 30%, 40% and 50% samples for $w < 60$. Apparently both models fit well the data. See figure 5 for a further comparison of data with these two fits.

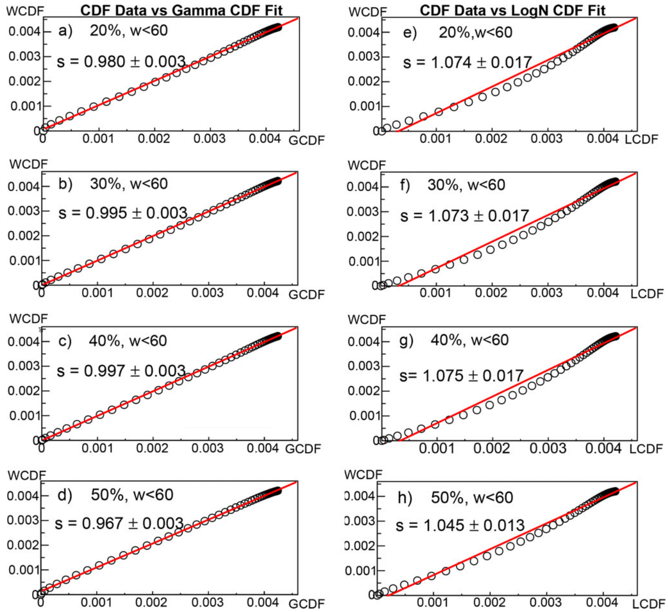


FIGURE 5
Panels a) to d) show the scatter plots of empirical WCDF vs theoretical fitted gamma CDF and panels e) to h) WCDF vs theoretical fitted log-normal CDF. A linear fit has been performed on each one of the different data samples. Corresponding chi-square of linear fits are shown in tables 2 and 3.

which compare data with fitted gamma and log-Normal models. This fitting approach of using data CDF instead of the histogram distribution has also the advantage of being bin size independent.

In order to further assess the quality of gamma and log-normal fits to GoL-generated wealth distribution, in figure 5 we show scatter plots of wealth CDF versus gamma CDF and versus log-normal CDF fits, denoted GCDF and LCDF respectively.

A linear fit is performed on these two scatter plots. Visually we appreciate that Gamma model fits much better the GoL generated wealth distribution data. In fact Log-Normal fit oscillates around and closer to data. Slope of linear fits and corresponding χ^2/ndf for gamma and log-normal models can be consulted in fifth and sixth columns of tables 2 and 3 respectively. These parameters confirm that gamma model fits better the data.

Fitting GoL-generated Wealth distribution: super-thermal income class

For the highest income sector of population, the GoL-generated Wealth distribution, is well fitted by the power law model, as figure 6 shows. Since the

% initial	Gamma, $0 < w < 60$				
alive cells	K_G	k	θ	s	χ^2/ndf
20%	0.9859 ± 0.0003	2.0702 ± 0.0012	6.5765 ± 0.0037	0.980×0.003	$6.9267 \times 10^{-8}/58$
30%	0.9843 ± 0.0003	2.4827 ± 0.0002	6.0142 ± 0.0002	0.995×0.003	$4.6381 \times 10^{-8}/58$
40%	0.9848 ± 0.0003	2.5083 ± 0.0013	6.0119 ± 0.0032	0.997×0.003	$4.4588 \times 10^{-8}/58$
50%	0.9850 ± 0.0003	2.3198 ± 0.0012	6.2060 ± 0.0033	0.977×0.003	$5.1392 \times 10^{-8}/58$

TABLE 2
Second to third columns show the values of gamma fit parameters of wealth distribution shown in figure 3 for the four data samples that are indicated in first column. Fifth column show the slope of the linear fit applied to the scatter plot of CDF GoL-generated wealth (WCDF) vs CDF of the fitted gamma model (GCDF) from figure 5. χ^2/ndf displayed value corresponds to the linear fit of WCDF vs GCDF shown in figure 5. Finally ndf means number of degree of fredom.

% initial	Log-Normal, $0 < w < 60$				
alive cells	K_{ln}	m	n	s	χ^2/ndf
20%	0.9172 ± 0.0003	2.4137 ± 0.0003	0.6927 ± 0.0002	1.074 ± 0.017	$1.4622 \times 10^{-6}/58$
30%	0.9283 ± 0.0003	2.5372 ± 0.0003	$0.6.3140 \pm 0.0002$	1.073 ± 0.017	$1.6356 \times 10^{-6}/58$
40%	0.9298 ± 0.0003	2.5460 ± 0.0003	0.6297 ± 0.00024	1.075 ± 0.017	$1.6753 \times 10^{-6}/58$
50%	0.9247 ± 0.0003	2.4885 ± 0.0003	0.6535 ± 0.0002	1.045 ± 0.013	$9.2614 \times 10^{-7}/58$

TABLE 3
Log-Normal fit parameters for figures 3 and 5. Again χ^2/ndf value corresponds to the linear fit to WCDF vs LCDF scatter plot from figure 5 and ndf means number of degree of fredom.

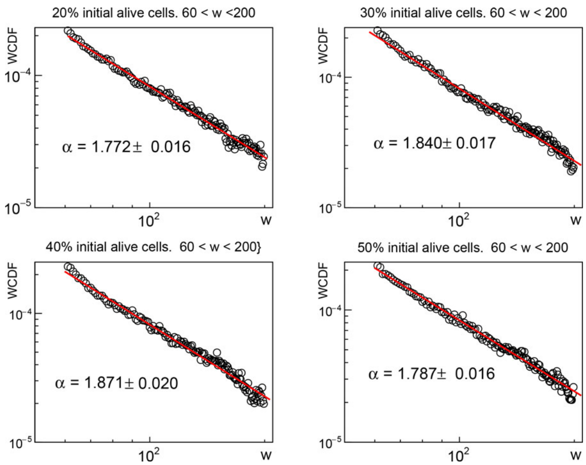


FIGURE 6
Power Law fits performed on tails of GoL-generated wealth distribution. Power law exponents are really negative as exponent in power law model M.4 indicates. Fits are performed on the corresponding CDF.

% initial alive cells	Power Law, $60 < w < 200$		
	K_{PL}	α	χ^2/ndf
20%	-1.246 ± 0.079	1.772 ± 0.016	0.5656 /137
30%	-0.931 ± 0.083	1.840 ± 0.017	0.6203 /137
40%	-0.789 ± 0.096	1.871 ± 0.020	0.8410 /137
50%	-1.170 ± 0.078	1.787 ± 0.016	0.5535 / 137

TABLE 4
Power Law fit parameters. Data samples, proportionality constant, Pareto exponent and χ^2/ndf are displayed in corresponding columns. The selected fitted region is $60 < w < 200$. Note that exponents indicated by formula of power law model M.4 are negative.

power law is very clear, the fit procedure is performed on the CDF points. As mentioned before, for the richest population, the selected fitted region of wealth distribution is $60 < w < 200$. The lower cut-off value, $w = 60$ as mentioned before, was selected by eye, because starting domain of the power law regime is clearly delimited in all wealth distributions plots displayed at figure 3 for all our data samples.

The upper cut off value $w = 200$ was selected to eliminate noise coming from periodic GoL patterns as explained above at section 2.1, as well as border size effects, visible for big values of wealth as a sudden and fast descending "step" at the end of the domain of plots of wealth distribution of same figure 3 and figure (1f). Perhaps we could select a higher than $w = 200$, upper cut off value, but since the power law signal distribution is very clear and covers a bit more than three quarters of the total domain of wealth distribution, this "safe" selection for the upper cut is enough good to evidence visually and statistically the emergence of a power law in the GoL generated wealth distribution, i.e. the emergence of a superthermal income class following [27] terminology. The values of Pareto exponents obtained by the fitting procedure showed at figure 6, that is a zoom in of figure 3 at the region $60 < w < 200$, are displayed in table 4.

4 GINI COEFFICIENT OF THE GOL GENERATED WEALTH DISTRIBUTION

The Gini index or Gini coefficient is an economic indicator that allows to quantify the inequality of a distribution of wealth, and is defined as:

$$G = \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} \left| x_i - x_j \right|}{2N^2 \mu} \qquad (2)$$

% initial alive cells:	**20%**	**30%**	**40%**	**50%**	**60%**
Gini:	0.428	0.384	0.381	0.399	0.472

TABLE 5
Gini Coefficients for each one of the four GoL generated samples. An additional Gini value for a sample with 60% of initial alive cells has been included.

Where N is the number of agents, μ is the mean value of the wealth distribution and x_i, $I = 1, \ldots, N$ is the wealth of the $i - th$ agent.

The Gini index varies from one to zero, being zero the case in which each individual of a population has the same amount of money and one when a single individual accumulates all the wealth of the system. Many countries and international institutions have used the Gini index, among other similar indexes, being possibly the world's most common indicator of inequality between rich and poor sectors of a country or society. Gini index has been measured throughout the world, ranging from 0.250 (Ukraine at 2016) to 0.571 (Zambia at 2015) [44].

We calculate the Gini index for the already analyzed data samples and one additional GoL-generated sample with an initial density of live cells of 60%. Results are shown in the Table 5, where one can see that these values are very close to those of the presented by the economic global indicators.

Lorenz curve of GoL-generated wealth distribution

The Lorenz curve is a graphical representation of the relative distribution of a population's income, the Gini index is the quotient between the areas generated by this curve and the equity line. Corresponding Lorenz curves of samples generated by GoL data can be consulted in figure 7.

5 DISCUSSION AND SUMMARY

In this paper we confirm that complexity of Conway's Game of Life cellular automaton is rich enough to reproduce all the universal statistical properties that are found in wealth distributions of real socio-economic systems. Unlike a simple, closed system with a fixed number of interacting particles that generates a Gibbs-Boltzmann (exponential) distribution by only reallocating wealth between their different agents (thermal income class), the wealth distribution generated by GoL shows an asymptotic power law decay corresponding to the wealth of the richest segment of the population, the sector where wealth is created (superthermal income class).

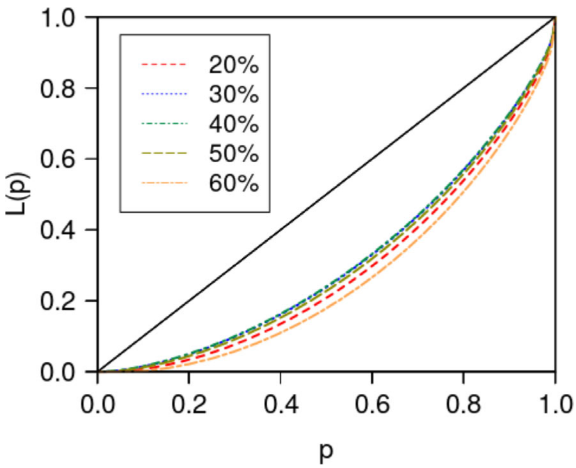


FIGURE 7
Lorenz curves for the analyzed data, the initial percentages of live cells are indicated. Again an additional Lorentz curve for a sample with 60% of initial alive cells has been included.

We show that medium and low segments of the distribution generated by GoL follow a gamma distribution. This distribution is observed in real socio-economic data describing the low-medium wealth sector of a population. Econophysics agent based models explain the origin of the gamma distribution as emerging from the the processes of reallocating wealth (energy) between the agents plus additionaly, the presence of savings [23].

Furthermore, the simultaneous emergence in GoL of the two income classes, thermal and superthermal as observed in socio-economic real data, is really a non-trivial emergent property than not many complex systems share.

In our opinion it is astonishing the way a very simple system, such as GoL is able to display and reproduce such a rich, complicated and realistic behavior. GoL automaton may be the simplest system capable of doing this.

As a final corollary, a fundamental explanation of the source of all empirical properties reported in this paper would be a truly interesting and worthwhile goal to achieve.

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REFERENCES

[1] Adamatzky A., editor. (2010). *Game of Life Cellular Automata*. Springer.

[2] Bylinking A A., Chemyavskaya N S., and Rostsovtsev A A. (2014). Hydrodinamic extention of a two-component model for hadroproduction in heavy-ion collisions. *Physical Review C*, 018201:018201–1–018201–4.

[3] Chatterjee A. and Chakrabarti B. K. (2007). Kinetic exchange models for income and wealth distributions. *European Physical Journal B*, 60:135–149.

[4] Chatterjee A., Yarlagadda S., and Chakrabarti B K. (2005). *Econophysics of Wealth Distributions: Econophys-Kolkata I*. Springer.

[5] Drăgulescu A. and Yakovenko V M. (2000). Statistical mechanics of money. *The European Physical Journal B*, 17:723–729.

[6] Drăgulescu A and Yakovenko V M. (2001). Exponential and power-law probability distributions of wealth and income in the united kingdom and the united states. *Physica A: Statistical Mechanics and its Applications*, 229:213–221.

[7] Hasegawa A., Mima K., and Duong-van M. (1985). Plasma distribution function in asuperthermal radiation field. *Physical Review Letters*, 54:2608–2610.

[8] Prinz A. (2017). Rankings of coordinations games: the dutch top 2000 pop songs rankings. *J. Cult. Econ.*, 41:379–401.

[9] Hernández-Montoya A.R. and H. Tapia McClung. (2019). A diffusion approach to the dynamics of conway's game of life (gol): Emergence of multiple power law fluctuation regimes. *Journal of Cellular automata*, 14(3-4):225–240.

[10] Düring B., Matthes D., and Toscani G. (Nov 2008). Kinetic equations modelling wealth redistribution: A comparison of approaches. *Phys. Rev. E*, 78:056103.

[11] Gini C. (1921). Measurement of inequality of incomes. *The Economic Journal*, 31:124–126.

[12] Tolman R C. (1938). *The Principles Of Statistical Mechanics*. Oxford At The Clarendon Press.

[13] Challet D. and Zhang Y-C. (1997). Emergence of cooperation and organization in an evolutionary game. *Physica A: Statistical Mechanics and its Applications*, 246(3-4):407–418.

[14] Samanidou E., Zschischang E., Stauffer D., and T. Lux. Microscopic models of financial markets. Economics Working Papers 2006-15. Christian-Albrechts-University of Kiel, Department of Economics.

[15] Coronel-Brizio H F. and Hernndez-Montoya A R. (2005). On fitting the pareto-levy distribution to stock market index data: selecting a suitable cut off value. *Physica A: Statistical Mechanics and its Applications*, 354:437–449.

[16] Schweitzer F. (2003). *Brownian Agents and Active Particles. On the Emergence of Complex Behavior in the Natural and Social Sciences*. Springer Series in Synergetics.

[17] Qiu G., Kandhai D., and Sloot P M A. (2007). Understanding the complex dynamics of stock markets through cellular automata. *Physical Review E*, 75(4):046116.

[18] Levy H., Levy M., and Solomon S. (2000). *Microscopic simulation of financial markets: from investor behavior to market phenomena*. Academic Press. Inc., Orlando, FL, USA.

[19] Desai M I., Mason G M., Dwyer J R., and et al. (2004). Evidence for a suprathermal seed population of heavy ions accelerated by interplanetary shocks near 1 au. *The Astrophysical Journal*, 588:1149–1162.

[20] Newman M E J. (2005). Power laws, pareto distributions and zipf's law. *Contemporary Physics*, 46(5):323–351.

[21] Von Neumann J. and Burks A W., editors. (1966). *Theory of self-reproducing automata*. University of Illinois Press Urbana. Urbana Champaign, IL, USA.

[22] Bouchaud J-P. and Potters M. (2009). Theory of financial risk and derivative pricing.

[23] Chakrabarti B K., Chakraborti A., Chakravarty S R., and Chatterjee A. (2013). *Econophysics of Income and Wealth Distributions*. Cambridge University Press.

[24] Bartolozzi M. and Thomas A W. (2004). Stochastic cellular automata model for stock market dynamics. *Physical review E*, 69(4):046112.

[25] Gardner M. (1970). Mathematical games: The fantastic combinations of John Conway's new solitaire game "life". *Scientific American*, 223(4):120–123.

[26] Mitzenmacher M. (2004). A brief history of generative models for power law and lognormal distributions. *Internet Mathematics*, 1(2):226–251.

[27] Yakovenko V M. and Sylva A C. (2005). Temporal evolution of the thermal and superthermal income classes in the usa during 1983-2001. *Europhysics Letters*, 69:304–310.

[28] Kaldor N. (1961). Capital accumulation and economic growth. In Hague D C., editor, *The Theory of Capital: Proceedings of a Conference held by the International Economic Association*, pages 177–222. Palgrave Macmillan UK, London.

[29] Mantegna R N. and Stanley H E. (1999). *Introduction to econophysics: correlations and complexity in finance*. Cambridge university press.

[30] Bak P., Tang C., and Wiesenfeld K. (1987). Self-organized criticality: An explanation of the $1/f$ noise. *Phys. Rev. Lett.*, 59:381–384.

[31] Brun R. and Rademakers F. (Sep 1996). Root - an object oriented data analysis framework. In *Proceedings AIHENP'96 Workshop*, Lausanne. Nucl. Inst. & Meth. in Phys. Res. A 389 (1997) 81-86.

[32] Collier M R. (2004). Are magnetospheric suprathermal particle distributions (**k** functions) inconsistent with maximum entropy considerations? *Advances in Space Research*, 33:2108–2112.

[33] Cont R. (2001). Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance*, 1(2):223–236.

[34] Hernández-Montoya A R., Coronel-Brizio H F., Stevens-Ramírez G A., Rodríguez-Achach M E., Politi M., and Scalas E. (2012). Emerging properties of financial time series in the "game of life". *Physical Review E*, 84(066104):066104–1–066104–9.

[35] Kunte S. and Gore P. (1993). The paradox of large samples. *Current Science*, 62:393–395.

[36] Ulam S. (1962). On some mathematical problems connected with patterns of growth of figures. *Proceedings of Symposia in Applied Mathematics*, 14:215–224.

[37] Wolfram S. (2002). *A New Kind of Science*. Wolfram Media.

[38] Lux T. and Marchesi M. (1998). Volatility clustering in financial markets: A microsimulation of interacting agents. *Discussion Paper Serie B*, 31(16):7–10.

[39] Zhou T., Zhou P-L., Wang B-H., Tang Z-N., and Liu J. (2004). Modeling stock market based on genetic cellular automata. *International Journal of Modern Physics B*, 18(17n19):2697–2702.

[40] Garibaldi U. and Scalas E. (2010). *Finitary Probabilistic Methods in Econophysics*. Cambridge University Press. Chapter 10, pgs 228 – 261.

[41] Pareto V. (1897). *Cours d'Économie Politique, Vol. I*. Lausanne, F. Rouge.

[42] Yakovenko V.M. and Barkley Roser J. Jr. (2009). Colloquium:statistical mechanics of money, wealth and income. *Reviews of Modern Physics*, 81:1703–1725.

[43] Montroll E. W. and Shlesinger M. F. (1982). On 1/f noise and other distributions with long tails. *Proceedings of the National Academy Sci USA*, 79:3380–3383.

[44] Development Research Group World Bank. World development indicators: Distribution of income or consumption. http://wdi.worldbank.org/table/1.3. Accessed December 26, 2018.