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Rough Sets and Rule Induction by an Approach Based on Coverings in Information Tables

MICHINORI NAKATA[1,*], NORIO SAITO[1], HIROSHI SAKAI[2] AND TAKESHI FUJIWARA[3]

[1]*Faculty of Management and Information Science, Josai International University, Japan*
[2]*Faculty of Engineering, Kyushu Institute of Technology, Japan*
[3]*Faculty of Informatics, Tokyo University of Information Sciences, Japan*

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Rough sets, which are a pair of lower and upper approximations, and rules induced from them are described by an approach using coverings in an information table with similarity of values. Lots of possible coverings on a set of attributes are derived in an information table with incomplete information, whereas only one covering is derived in an information table with complete information. New difficulty due to computational complexity is not caused in any information table with incomplete information because of the lattice structure that the family of possible coverings has. Twofold rough sets are derived, which consist of certain rough sets and possible rough sets, using only the minimum and maximum possible coverings. These two possible coverings are obtained from the minimum and the maximum possible indiscernibility relations which are equal to the intersection and the union of indiscernibility relations derived from possible tables. Four kinds of rules with accuracy and support are induced from the twofold rough sets. The computational complexity for the number of objects in incomplete information tables is the same as in complete information tables.

Keywords: Twofold rough sets, Rule induction, Incomplete information, Coverings, Indiscernibility relations

1 INTRODUCTION

Rough sets by Pawlak [1] classify objects into granules by using equality of data characterizing them. The rough sets are well known as an applicable tool

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* Corresponding author: E-mail: nakatam@ieee.org

to inducing rules from data in various fields [2, 3]. The traditional rough sets are typically used when complete information is obtained and data similarities do not have to be considered. Data similarities, however, are common in the real world. Furthermore, data incompleteness appears everywhere in the real world. Thus, it is not sufficient for information processing of the actual data unless we handle similar data and incompletely represented data.

For data with similarities, the degree of similarity is used under a threshold [4]. A restriction such as min-max transitivity should not be imposed on the degree, because the restriction is contrary to human intuition [5]. The degree of similarity is asymmetric, because the degree of object o which is similar to object o' is not always the same as the degree of o' which is similar to o.

When incompletely represented data is obtained, there are four approaches to handling it. The first is to replace the data by a plausible value. The plausible value is specified by statistical methods [6]. As a simple example, the most frequently occurring value, the average value and so on are used. The second is to give the data equality without touching the data itself, which was proposed by Kryszkiewicz [7] and afterwards was extended [8,9]. The third is to use maximal consistent blocks [10], where the data is contained in blocks. The three approaches keep only one possibility among lots of possibilities of incompletely represented data. This means that information loss will occur. This information loss causes poor results [11, 12].

The other approach is to apply possible world semantics to an information table with incompletely obtained data, which was proposed by Lipski [13] in the field of incomplete databases. The approach deals with all possibilities that any incompletely obtained data has. No information loss occurs. We, therefore, adopt this approach.

We develop rough sets and rule induction using possible indiscernibility relations. A possible indiscernibility relation is a possible world in possible world semantics in our approach, although a possible table is a possible world in Lipski. Categorical values can be handled using possible tables [14,15], but continuous values cannot because the number of possible tables is infinite for the continuous values [16]. To use possible indiscernibility relations means that the categorical values and the continuous values are handled in the same framework.

The structure of the paper is as follows. In the next section, an approach using the covering obtained from the indiscernibility relation on a set of attributes is described to derive rough sets and rules in a complete information table. The subsequent section develops the approach in an incomplete information table under possible world semantics. The fourth section describes algorithms of obtaining rough sets and rules in this approach. The last section describes conclusions.

2 ROUGH SETS AND RULE INDUCTION USING COVERINGS IN COMPLETE INFORMATION TABLES

A complete information table is expressed with three components: U, $V(=\cup_{a\in AT} V(a))$, and AT which are a set of objects, called the universe, the value set of attribute a, and a set of attributes where $a \in AT : U \rightarrow V(a)$, respectively. Indiscernibility relation $I_A^{\delta *}$ meaning that objects are indiscernible on set $A \subseteq AT$ of attributes under threshold δ_A is:

$$I_A^\delta = \{(o, o') \in U \times U \mid SIM_A(o, o') \geq \delta_A\}, \quad (1)$$

$$= \cap_{a\in A} I_a^\delta, \quad (2)$$

where $SIM_A(o, o')$ expresses at what degree objects o and o' are similar similar in terms of A and δ_A is the similarity threshold for the values that A takes and is $(\delta_{a_1}, \ldots, \delta_{a_l})$ when $A = \{a_1, \ldots, a_l\}$.

$$SIM_A(o, o') = \min_{a\in A} SIM_a(o, o'), \quad (3)$$

$$SIM_a(o, o') = sim(a(o), a(o')), \quad (4)$$

where $sim(a(o), a(o'))$ is the similarity degree of attribute value $a(o)$ to $a(o')$, which is given by experts such that it is reflexive, asymmetric, and not transitive.

Proposition 1. *If $\delta 1_A \geq \delta 2_A$, then $I_A^{\delta 1} \subseteq I_A^{\delta 2}$, where $\delta 1_A \geq \delta 2_A$ is $\forall a \in A\ \delta 1_a \geq \delta 2_a$.*

Proof. Straightforward.

Under I_A^δ, indiscernible class $K(o)_A^{\delta \dagger}$ of o on A is expressed in:

$$K(o)_A^\delta = \{o' \mid (o, o') \in I_A^\delta\}, \quad (5)$$

$$= \cap_{a\in A} K(o)_a^\delta. \quad (6)$$

Family $\mathcal{K}_A^\delta$ of indiscernible classes on A is:

$$\mathcal{K}_A^\delta = \cup_{o\in U}\{K(o)_A^\delta\}. \quad (7)$$

* Unless confusion may arise, A and/or δ are omitted for symbols.

† This class is not a tolerance class. See [17, 18] for tolerance classes.

Clearly, $\cup_{K \in \mathcal{K}_A} K = U$. According to Zakowski [19], $\mathcal{K}_A$ is a covering, which is unique for A. Under $\mathcal{K}_A$, minimal description $Md\mathcal{K}_A(o)$ of object o, formulated by [20], is:

$$Md\mathcal{K}_A(o) = \\ \{K \in \mathcal{K}_A \mid o \in K \wedge \forall K' \in \mathcal{K}_A(o \in K' \wedge K' \subseteq K \Rightarrow K = K')\}. \quad (8)$$

Set $CFriend_{\mathcal{K}_A}(o)$ of close friends of o with respect to $\mathcal{K}_A$, proposed by [21], is:

$$CFriend_{\mathcal{K}_A}(o) = \cup_{K \in Md\mathcal{K}_A(o)} K. \quad (9)$$

Maximal description $MD\mathcal{K}_A(o)$ of object o, described by [21,22], is:

$$MD\mathcal{K}_A(o) = \\ \{K \in \mathcal{K}_A \mid o \in K \wedge \forall K' \in \mathcal{K}_A(o \in K' \wedge K' \supseteq K \Rightarrow K = K')\}. \quad (10)$$

Example 1. *Let complete information table CT be denoted in TABLE 1. The universe consists of six objects $1, \ldots, 6$. a_1 and a_2 are attributes. Let*

U	CT: a_1	a_2
1	f	y
2	a	z
3	c	y
4	e	w
5	b	z
6	d	x

TABLE 1
Complete information table CT

similarity degree $sim(v, v')$ on $V(a_1) = \{a, b, c, d, e, f\}$ be

$$sim(v, v') = \begin{pmatrix} 1 & 0.2 & 0.7 & 0.6 & 0.2 & 0.3 \\ 0.3 & 1 & 0.7 & 0.7 & 0.2 & 0.4 \\ 0.9 & 0.7 & 1 & 0.2 & 0.3 & 0.3 \\ 0.9 & 0.8 & 0.2 & 1 & 0.7 & 0.7 \\ 0.1 & 0.2 & 0.3 & 0.7 & 1 & 0.9 \\ 0.3 & 0.4 & 0.3 & 0.7 & 0.7 & 1 \end{pmatrix}.$$

Indiscernibility relation I_{a_1} under $\delta_{a_1} = 0.75$ is:

$$I_{a_1} = \{(1,1), (2,2), (3,2), (3,3), (4,1), (4,4), (5,5), (6,2), (6,5), (6,6)\}.$$

Indiscernible classes that make up covering $\mathcal{K}_{a_1}$ on a_1 are:

$$\begin{aligned} K(1)_{a_1} &= \{1\}, K(2)_{a_1} = \{2\}, K(3)_{a_1} = \{2,3\}, \\ K(4)_{a_1} &= \{1,4\}, K(5)_{a_1} = \{5\}, K(6)_{a_1} = \{2,5,6\}. \end{aligned}$$

Under these indiscernible classes, covering $\mathcal{K}_{a_1}$ is:

$$\mathcal{K}_{a_1} = \{\{1\}, \{2\}, \{2,3\}, \{1,4\}, \{5\}, \{2,5,6\}\}.$$

Under covering $\mathcal{K}_A$, lower approximation $\underline{apr}_A(\mathcal{T})$ and upper approximation $\overline{apr}_A(\mathcal{T})$ of target set $\mathcal{T}$ for A are:

$$\underline{apr}_A(\mathcal{T}) = \{o \in U \mid K(o) \in \mathcal{K}_A \wedge K(o) \subseteq \mathcal{T}\}, \tag{11}$$
$$\overline{apr}_A(\mathcal{T}) = \{o \in U \mid K(o) \in \mathcal{K}_A \wedge K(o) \cap \mathcal{T} \neq \emptyset\}. \tag{12}$$

Proposition 2. *If $\delta 1_A \geq \delta 2_A$ on A, then $\underline{apr}_A^{\delta 1}(\mathcal{T}) \supseteq \underline{apr}_A^{\delta 2}(\mathcal{T})$ and $\overline{apr}_A^{\delta 1}(\mathcal{T}) \subseteq \overline{apr}_A^{\delta 2}(\mathcal{T})$, where $\delta 1_A \geq \delta 2_A$ is $\forall a \in A$ $\delta 1_a \geq \delta 2_a$ and the approximations with superscripts $\delta 1$ and $\delta 2$ show ones under thresholds $\delta 1$ and $\delta 2$, respectively.*

Proof. Straightforward from $\forall o$ $K(o)_A^{\delta 1} \subseteq K(o)_A^{\delta 2}$ under $\delta 1_A \geq \delta 2_A$,

Example 2. *Let target set $\mathcal{T}$ be specified by restriction $a_2 = z$, which is denoted by $\mathcal{T}_{a_2=z}$. $\mathcal{T}_{a_2=z} = \{2,5\}$ is obtained in information table CT of Example 1. Based on formulae (11) and (12) under the covering in Example 1, lower and upper approximations are:*

$$\begin{aligned} \underline{apr}_{a_1}(\mathcal{T}_{a_2=z}) &= \{2,5\}, \\ \overline{apr}_{a_1}(\mathcal{T}_{a_2=z}) &= \{2,3,5,6\}. \end{aligned}$$

The following rules are obtained from the lower and upper approximations. Let target set $\mathcal{T}_R$ be determined by restriction R.

- When $o \in \underline{apr}_A(\mathcal{T}_R)$, rule $A = A(o) \rightarrow R$ holds consistently: namely, its accuracy is 1 and its support is $|K(o)_A|/|U|$.
- When $o \in (\overline{apr}_A(\mathcal{T}_R) \backslash \underline{apr}_A(\mathcal{T}_R))$, rule $A = A(o) \rightarrow R$ holds inconsistently with accuracy $|K(o)_A \cap \mathcal{T}_R|/|K(o)_A|$ and support $|K(o)_A \cap \mathcal{T}_R|/|U|$.

Example 3. *Let go back to Example 2. Objects 2 and 5 are included in $\underline{apr}_{a_1}(\mathcal{T}_{a_2=z})$. Two rules are derived. Rule $a_1 = a \rightarrow a_2 = z$ holds* *consistently with accuracy 1 and support 1/6 and rule $a_1 = b \rightarrow a_2 = z$ holds consistently with accuracy 1 and support 1/6. Objects 3 and 6 are included in $(\overline{apr}_{a_1}(\mathcal{T}_{a_2=z}) \backslash \underline{apr}_{a_1}(\mathcal{T}_{a_2=z}))$. Two rules are derived. Rule $a_1 = c \rightarrow a_2 = z$ holds inconsistently with accuracy 1/2 and support 1/6 and rule $a_1 = d \rightarrow a_2 = z$ holds inconsistently with accuracy 2/3 and support 1/3.*

3 ROUGH SETS AND RULE INDUCTION USING POSSIBLE COVERINGS IN INCOMPLETE INFORMATION TABLES

An incompletely obtained value is expressed in a disjunctive set of possible value, where the missing value of attribute a is expressed with $\{v \mid v \in V(a)\}$.

There are lots of possible coverings derived from an incomplete information table [23, 24]. Despite this some authors deal with only one covering [25, 26], which is questionable. One possible indiscernibility relation creates one possible covering. We obtain a lot of possible indiscernibility relations on set A of attributes in an incomplete information table. FPI_A, the family of possible indiscernibility relations, is:

$$FPI_A = \{PI \mid PI = SI_A \cup (o, o') \wedge (o, o') \in \mathcal{P}(MPI_A \backslash SI_A)\}, \qquad (13)$$

where PI is a possible indiscernibility relation and $\mathcal{P}(MPI_A \backslash SI_A)$ denotes the power set of $MPI_A \backslash SI_A$.

$$MPI_A = \{(o, o') \in U \times U \mid \forall a \in A \exists v \in a(o) \exists v' \in a(o') sim(v, v') \geq \delta_a\}, \qquad (14)$$

$$SI_A = \{(o, o') \in U \times U \mid (o = o') \vee (\forall a \in A \forall v \in a(o) \forall v' \in a(o') sim(v, v') \geq \delta_a)\}, \qquad (15)$$

where pair $(o, o') \in SI_A$ is called a certain one on A, whereas pair $(o, o') \in (MPI_A \backslash SI_A)$ a possible one on A.

Proposition 3. *FPI_A constitutes the lattice based on set inclusion where $PI_{A,min}$, the minimum element, is equal to SI_A and $PI_{A,max}$, the maximum element, is equal to MPI_A.*

Proof. Straightforward from formulae (13)–(15).

Proposition 4. *$PI_{A,min} = \cap PI_A$ and $PI_{A,max} = \cup PI_A$ where PI_A is a possible indiscernibility relation on A.*

Proof. Straightforward from formula (13).

Let FPT_A be the family of possible tables on A.

$$FPT_A = \{PT \mid \forall o \in U\ \forall a \in A\ a(o)_{PT} = e \wedge e \in a(o)_{IT}\}, \quad (16)$$

where PT is a possible table and IT is an incomplete information table, and $a(o)_{PT}$ and $a(o)_{IT}$ are values of attribute a in PT and IT, respectively. A possible indiscernibility relation does not always correspond to a possible table, but the following propositions hold.

Proposition 5. *If $PTI_A \in FPTI_A$, then $PTI_A \in FPI_A$, where PTI_A is a indiscernibility relation derived from a possible table on A of incomplete information table IT and $FPTI_A$ is the family of PTI_A.*

Proof. Let PT be a possible table on A of incomplete information table IT. Indiscernibility relation PTI_A of PT is expressed in $\{(o, o') \in U \times U \mid \forall a \in A\ sim(a(o)_{PT}, a(o')_{PT}) \geq \delta_a\}$ from formulae (1). (3), and (4). $a(o)_{PT} \in a(o)_{IT}$ and $a(o')_{PT} \in a(o')_{IT}$. Thus, $PTI_A \in FPI_A$ from formulae (13)–(15).

Proposition 6. *$PI_{A,min} = \cap PTI_A,\ PI_{A,max} = \cup PTI_A$, where $PI_{A,min}$ and $PI_{A,max}$ are the minimum and the maximum possible indiscernibility relations, and PTI_A is the indiscernibility relation derived from a possible table PT_A using formulae (1), (3), and (4).*

Proof. $PI_{A,min}$ consists of the certain pairs where two object are certainly indiscernible. If $(o, o') \in PI_{A,min}$, $(o, o') \in SI_A$. This means $\forall PI_A \in FPI_A$ $(o, o') \in PI_A$ from formula (13). From proposition 5, if $PTI_A \in FPTI_A$, then $PTI_A \in FPI_A$. Thus, $(o, o') \in \cap PTI_A$ from proposition 5. If $(o, o') \in \cap PTI_A$, $\forall PTI_A \in FPTI_A$ $(o, o') \in PTI_A$. This means that $(o = o') \vee (\forall a \in A \forall v \in a(o) \forall v' \in a(o') sim(v, v') \geq \delta_a)$ is satisfied with (o, o'); namely, $(o, o') \in SI_A$. Thus, $(o, o') \in PI_{A,min}$.
If $(o, o') \in PI_{A,max}$, $(o, o') \in MPI_A$. (o, o') satisfies $\forall a \in A \exists v \in a(o) \exists v' \in a(o') sim(v, v') \geq \delta_a$ of formula (14). This means $\exists PTI_A$ $(o, o') \in PTI_A$. Thus, $(o, o') \in \cup PTI_A$. If $(o, o') \in \cup PTI_A$, $\forall a \in A \exists v \in a(o) \exists v' \in a(o') sim(v, v') \geq \delta_a$ is satisfied with (o, o'). This means $\exists PI_A$ $(o, o') \in PI_A$. Thus, $(o, o') \in PI_{A,max}$.

Example 4. *Let incomplete information table IT be denoted in TABLE 2. $\{b, e\}$ is the disjunctive set that means b or e.*

U	a_1	a_2
1	$\{a\}$	$\{y\}$
2	$\{b, e\}$	$\{z\}$
3	$\{a\}$	$\{w, y\}$
4	$\{d\}$	$\{x, z\}$
5	$\{c, f\}$	$\{z\}$

TABLE 2
Incomplete Information Table IT

Under formula (16), four possible tables PT_1, PT_2, PT_3 and PT_4 on attribute a_1 are derived from incomplete information table IT, as is shown in TABLE 3.

PT_1

U	a_1	a_2
1	$\{a\}$	$\{y\}$
2	$\{b\}$	$\{z\}$
3	$\{a\}$	$\{w, y\}$
4	$\{d\}$	$\{x, z\}$
5	$\{c\}$	$\{z\}$

PT_2

U	a_1	a_2
1	$\{a\}$	$\{y\}$
2	$\{e\}$	$\{z\}$
3	$\{a\}$	$\{w, y\}$
4	$\{d\}$	$\{x, z\}$
5	$\{c\}$	$\{z\}$

PT_3

U	a_1	a_2
1	$\{a\}$	$\{y\}$
2	$\{b\}$	$\{z\}$
3	$\{a\}$	$\{w, y\}$
4	$\{d\}$	$\{x, z\}$
5	$\{f\}$	$\{z\}$

PT_4

U	a_1	a_2
1	$\{a\}$	$\{y\}$
2	$\{e\}$	$\{z\}$
3	$\{a\}$	$\{w, y\}$
4	$\{d\}$	$\{x, z\}$
5	$\{f\}$	$\{z\}$

TABLE 3
Four possible tables PT_1, PT_2, PT_3 and PT_4 derived from IT on a_1

Let $sim(u, v)$ be the same in Example 1. Under $\delta_{a_1} = 0.75$, set SI_{a_1} of certain pairs is derived from IT using formula (15):

$$\{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 1), (4, 3), (4, 4), (5, 5)\}.$$

Set $MPI_{a_1} \backslash SI_{a_1}$ of possible pairs is derived using formulae (14) and (15):

$$\{(2, 5), (4, 2), (5, 1), (5, 3)\}.$$

Under formulae (13) – (15), family FPI_{a_1} of possible indiscernibility relations is:

$$FPI_{a_1} = \{PI_1, \cdots, PI_{16}\}.$$

The sixteen possible indiscernibility relations are:

$$\begin{aligned}
PI_1 &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5)\},\\
PI_2 &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5),\\
&\quad (2,5)\},\\
PI_3 &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5),\\
&\quad (4,2)\},\\
PI_4 &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5),\\
&\quad (5,1)\},\\
PI_5 &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5),\\
&\quad (5,3)\},\\
PI_6 &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5),\\
&\quad (2,5),(4,2)\},\\
PI_7 &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5),\\
&\quad (2,5),(5,1)\},\\
PI_8 &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5),\\
&\quad (2,5),(5,3)\},\\
PI_9 &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5),\\
&\quad (4,2),(5,1)\},\\
PI_{10} &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5),\\
&\quad (4,2),(5,3)\},\\
PI_{11} &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5),\\
&\quad (5,1),(5,3)\},\\
PI_{12} &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5),\\
&\quad (2,5),(4,2),(5,1)\},\\
PI_{13} &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5),\\
&\quad (2,5),(4,2),(5,3)\},\\
PI_{14} &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5),\\
&\quad (2,5),(5,1),(5,3)\},\\
PI_{15} &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5),
\end{aligned}$$

$$
\begin{aligned}
& (4,2),(5,1),(5,3)\}, \\
PI_{16} &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5), \\
& (2,5),(4,2),(5,1),(5,3)\}.
\end{aligned}
$$

Indiscernibility relations PTI_1, PTI_2, PTI_3, and PTI_4 derived from possible tables PT_1, PT_2, PT_3, and PT_4, respectively, are:

$$
\begin{aligned}
PTI_1 &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5), \\
& (4,2),(5,1),(5,3)\}, \\
PTI_2 &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5), \\
& (5,1),(5,3)\}, \\
PTI_3 &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5), \\
& (4,2)\}, \\
PTI_4 &= \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,1),(4,3),(4,4),(5,5), \\
& (2,5)\}.
\end{aligned}
$$

Relationship between the possible indiscernibility relations and the indiscernibility relations from possible tables is represented by $PTI_1 = PI_{15}$, $PTI_2 = PI_{11}$, $PTI_3 = PI_3$, and $PTI_4 = PI_2$; namely, four possible indiscernibility relations PI_2, PI_3, PI_{11}, and PI_{15} have the corresponding possible tables. PI_1 and PI_{16} are the minimum and the maximum possible indiscernibility relations, respectively. The relationships represented by $PI_1 = \cap_{i=1,4} PTI_i$ and $PI_{16} = \cup_{i=1,4} PTI_i$ hold.

Possibly indiscernible class $PK(o)_{A,j}$ of object o in $PI_j \in FPI_A$ is:

$$
\begin{aligned}
PK(o)_{A,j} &= \{o' \mid (o,o') \in PI_j \wedge PI_j \in FPI_A\}, && (17) \\
&= \cap_{a \in A} PK(o)_{a,j}. && (18)
\end{aligned}
$$

$PK(o)_{A,min}$, the minimum possibly indiscernible class on A, and $PK(o)_{A,max}$, the maximum possibly indiscernible class, are:

$$
\begin{aligned}
PK(o)_{A,min} &= \{o' \mid (o,o') \in PI_{A,min}\}, && (19) \\
&= \cap_{j=1,h} \{o' \mid (o,o') \in PI_j \wedge PI_j \in FPI_A\}, && (20) \\
PK(o)_{A,max} &= \{o' \mid (o,o') \in PI_{A,max}\}, && (21) \\
&= \cup_{j=1,h} \{o' \mid (o,o') \in PI_j \wedge PI_j \in FPI_A\}, && (22)
\end{aligned}
$$

where h is the number of possible indiscernibility relations. Under formula (18),

$$PK(o)_{A,min} = \cap_{a \in A} PK(o)_{a,min}, \quad (23)$$
$$PK(o)_{A,max} = \cap_{a \in A} PK(o)_{a,max}. \quad (24)$$

Proposition 7. *If $PI_{A,k} \subseteq PI_{A,l}$, then $\forall o \in U$ $PK(o)_{A,k} \subseteq PK(o)_{A,l}$.*

Proof. If $(o, o') \in PI_{A,k}$, $o' \in PK(o)_{A,k}$ from formula (17). $(o, o') \in PI_{A,l}$ from $PI_{A,k} \subseteq PI_{A,l}$, and also $o' \in PK(o)_{A,l}$ from formula (17). Thus, this proposition holds.

This proposition means that the family of possibly indiscernible classes for any object also has a lattice structure for set inclusion with the minimum and maximum elements. The minimum element for object o, the minimum possibly indiscernible class of o, is $PK(o)_{A,min}$ derived from $PI_{A,min}$. The maximum element for o, the maximum possibly indiscernible class of o, is $PK(o)_{A,max}$ derived from $PI_{A,max}$.

Example 5. *Possibly indiscernible classes, minimum possibly indiscernible classes, and maximum possibly indiscernible classes on a_1 of objects are derived applying formulae (17), (20), and (22) to possible indiscernibility relations of Example 4. For object* 1*,*

$$PK(1)_{a_1,j} = \{1, 3\} \ for \ j = 1, \ldots, 16,$$
$$PK(1)_{a_1,min} = \{1, 3\}, PK(1)_{a_1,max} = \{1, 3\}.$$

For object 2*,*

$$PK(2)_{a_1,j} = \{2\} \ for \ j = 1, 3, 4, 5, 9, 10, 11, 15,$$
$$PK(2)_{a_1,j} = \{2, 5\} \ for \ j = 2, 6, 7, 8, 12, 13, 14, 16,$$
$$PK(2)_{a_1,min} = \{2\}, PK(2)_{a_1,max} = \{2, 5\}.$$

For object 3*,*

$$PK(3)_{a_1,j} = \{1, 3\} \ for \ j = 1, \ldots, 16,$$
$$PK(3)_{a_1,min} = \{1, 3\}, PK(3)_{a_1,max} = \{1, 3\}.$$

For object 4,

$$
\begin{aligned}
PK(4)_{a_1,j} &= \{1,3,4\}\ for\ j = 1,2,4,5,7,8,11,14,\\
PK(4)_{a_1,j} &= \{1,2,3,4\}\ for\ j = 3,6,9,10,12,13,15,16,\\
PK(4)_{a_1,min} &= \{1,3,4\},\ PK(4)_{a_1,max} = \{1,2,3,4\}.
\end{aligned}
$$

For object 5,

$$
\begin{aligned}
PK(5)_{a_1,j} &= \{5\}\ for\ j = 1,2,3,6,\\
PK(5)_{a_1,j} &= \{1,5\}\ for\ j = 4,7,9,12,\\
PK(5)_{a_1,j} &= \{3,5\}\ for\ j = 5,8,10,13,\\
PK(5)_{a_1,j} &= \{1,3,5\}\ for\ j = 11,14,15,16,\\
PK(5)_{a_1,min} &= \{5\},\ PK(5)_{a_1,max} = \{1,3,5\}.
\end{aligned}
$$

One possible covering is derived from one possible indiscernibility relation. $P\mathcal{K}_{A,j}$, the possible covering obtained from possible indiscernibility relation $PI_{A,j}$, is:

$$P\mathcal{K}_{A,j} = \cup_{o \in U}\{PK(o)_{A.j}\}. \tag{25}$$

Proposition 8. *Family $FP\mathcal{K}_A$ of possible coverings is a lattice for $\sqsubseteq$ where $\sqsubseteq$ is expressed as $FE \sqsubseteq FE'$ if $\forall E \in FE\ \exists E' \in FE' \wedge E \subseteq E'$.*

Proof. Straightforward from Proposition 7.

Minimum possible covering $P\mathcal{K}_{A,min}$ on A and maximum possible covering $P\mathcal{K}_{A,max}$ are:

$$
\begin{aligned}
P\mathcal{K}_{A,min} &= \cup_{o \in U}\{PK(o)_{A,min}\}, &(26)\\
&= \cup_{o \in U}\{\cap_{i=1,h} PK(o)_{A,i}\}, &(27)\\
P\mathcal{K}_{A,max} &= \cup_{o \in U}\{PK(o)_{A,max}\}, &(28)\\
&= \cup_{o \in U}\{\cup_{i=1,h} PK(o)_{A,i}\}, &(29)
\end{aligned}
$$

where h is the number of possible coverings.

Using formulae (8)–(10), we can obtain the family of possible minimum description, the family of possible sets of close friends, and the family of possible maximum descriptions from the family of possible coverings. The family of possible maximum descriptions keeps a lattice structure, but the family of possible minimum description and the family of possible sets of close friends do not.

Example 6. *Under formula (17), possibly indiscernible classes in each possible indiscernibility relation* $PI_{a_1,j}$ *with* $j = 1, \dots, 16$ *are obtained from example 5. For example, in* $PI_{a_1,1}$, $PK(1)_{a_1,1} = \{1, 3\}$, $PK(2)_{a_1,1} = \{2\}$, $PK(3)_{a_1,1} = \{1, 3\}$, $PK(4)_{a_1,1} = \{1, 3, 4\}$, $PK(5)_{a_1,1} = \{5\}$. *In* $PI_{a_1,16}$, $PK(1)_{a_1,16} = \{1, 3\}$, $PK(2)_{a_1,16} = \{2, 5\}$, $PK(3)_{a_1,16} = \{1, 3\}$, $PK(4)_{a_1,16} = \{1, 2, 3, 4\}$, $PK(5)_{a_1,16} = \{1, 3, 5\}$.
Under formula (25), each possible covering $P\mathcal{K}_{a_1,j}$ *with* $j = 1, \dots, 16$ *is obtained as follows:*

$$
\begin{aligned}
P\mathcal{K}_{a_1,1} &= \{\{1,3\},\{2\},\{1,3,4\},\{5\}\},\\
P\mathcal{K}_{a_1,2} &= \{\{1,3\},\{2,5\},\{1,3,4\},\{5\}\},\\
P\mathcal{K}_{a_1,3} &= \{\{1,3\},\{2\},\{1,2,3,4\},\{5\}\},\\
P\mathcal{K}_{a_1,4} &= \{\{1,3\},\{2\},\{1,3,4\},\{1,5\}\},\\
P\mathcal{K}_{a_1,5} &= \{\{1,3\},\{2\},\{1,3,4\},\{3,5\}\},\\
P\mathcal{K}_{a_1,6} &= \{\{1,3\},\{2,5\},\{1,2,3,4\},\{5\}\},\\
P\mathcal{K}_{a_1,7} &= \{\{1,3\},\{2,5\},\{1,3,4\},\{1,5\}\},\\
P\mathcal{K}_{a_1,8} &= \{\{1,3\},\{2,5\},\{1,3,4\},\{3,5\}\},\\
P\mathcal{K}_{a_1,9} &= \{\{1,3\},\{2\},\{1,2,3,4\},\{1,5\}\},\\
P\mathcal{K}_{a_1,10} &= \{\{1,3\},\{2\},\{1,2,3,4\},\{3,5\}\},\\
P\mathcal{K}_{a_1,11} &= \{\{1,3\},\{2\},\{1,3,4\},\{1,3,5\}\},\\
P\mathcal{K}_{a_1,12} &= \{\{1,3\},\{2,5\},\{1,2,3,4\},\{1,5\}\},\\
P\mathcal{K}_{a_1,13} &= \{\{1,3\},\{2,5\},\{1,2,3,4\},\{3,5\}\},\\
P\mathcal{K}_{a_1,14} &= \{\{1,3\},\{2,5\},\{1,3,4\},\{1,3,5\}\},\\
P\mathcal{K}_{a_1,15} &= \{\{1,3\},\{2\},\{1,2,3,4\},\{1,3,5\}\},\\
P\mathcal{K}_{a_1,16} &= \{\{1,3\},\{2,5\},\{1,2,3,4\},\{1,3,5\}\}.
\end{aligned}
$$

Family $FP\mathcal{K}_{a_1}$ *has the lattice structure shown in FIGURE 1.*

Relationship between the minimum and maximum possible coverings and coverings from possible tables is represented by:

$$
\begin{aligned}
P\mathcal{K}_{A,min} &= \cup_{o\in U}\{\cap_{i=1,h}K(o)_{A,i}\}, && (30)\\
P\mathcal{K}_{A,max} &= \cup_{o\in U}\{\cup_{i=1,h}K(o)_{A,i}\}, && (31)
\end{aligned}
$$

where $K(o)_{A,i}$ is the class of o in the covering derived from $PTI_{A,i}$ using formulae (5) and (7) and h is the number of possible tables.

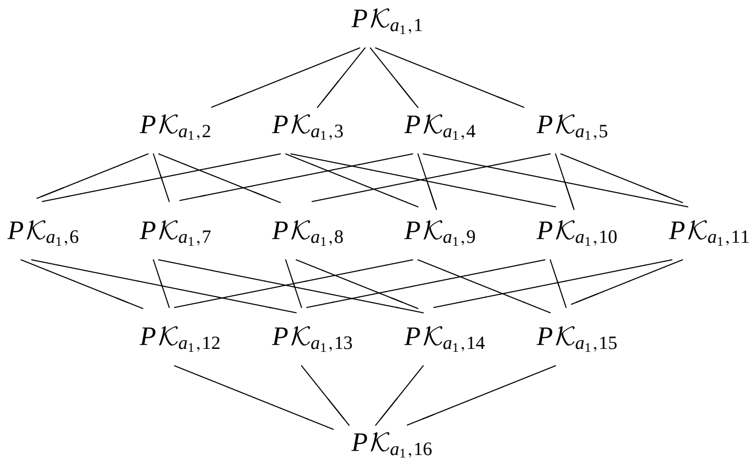

FIGURE 1
Lattice structure of family $FP\mathcal{K}_{a_1}$ of possible covering

Under possible covering $P\mathcal{K}_{A,j}$, two approximations of target set $\mathcal{T}$ are:

$$\underline{apr}_{A,j}(\mathcal{T}) = \{o \in U \mid PK(o) \subseteq \mathcal{T} \wedge PK(o) \in P\mathcal{K}_{A,j}\}, \quad (32)$$
$$\overline{apr}_{A,j}(\mathcal{T}) = \{o \in U \mid PK(o) \cap \mathcal{T} \neq \emptyset \wedge PK(o) \in P\mathcal{K}_{A,j}\}. \quad (33)$$

Proposition 9. *If $P\mathcal{K}_k \sqsubseteq P\mathcal{K}_l$ for possible indiscernibility coverings $P\mathcal{K}_k, P\mathcal{K}_l \in FP\mathcal{K}_A$, then $\underline{apr}_{A,k}(\mathcal{T}) \supseteq \underline{apr}_{A,l}(\mathcal{T})$ and $\overline{apr}_{A,k}(\mathcal{T}) \subseteq \overline{apr}_{A,l}(\mathcal{T})$.*

Proof. If $P\mathcal{K}_k \sqsubseteq P\mathcal{K}_l$, $\forall PK \in P\mathcal{K}_k \exists PK' \in P\mathcal{K}_l \; PK \subseteq PK'$. Thus, this proposition holds.

Proposition 9 shows that the families of approximations are also lattices.

Example 7. *Under $\mathcal{T} = \{2, 5\}$, for example, applying formulae (32) and (33) to $P\mathcal{K}_{a_1,1}$ and $P\mathcal{K}_{a_1,16}$, we obtain $\underline{apr}_{a_1,1}(\mathcal{T}) = \{2, 5\}$, $\underline{apr}_{a_1,16}(\mathcal{T}) = \{2\}$, $\overline{apr}_{a_1,1}(\mathcal{T}) = \{2, 5\}$, and $\overline{apr}_{a_1,16}(\mathcal{T}) = \{2, 4, 5\}$.*

Aggregating approximations in each possible covering, we obtain four approximations: certain lower approximation $S\underline{apr}_A(\mathcal{T})$ of $\mathcal{T}$, possible lower approximation $P\underline{apr}_A(\mathcal{T})$, certain upper approximations $S\overline{apr}_A(\mathcal{T})$, and possible upper approximation $P\overline{apr}_A(\mathcal{T})$.

$$
\begin{aligned}
S\underline{apr}_A(\mathcal{T}) &= \{o \in U \mid \forall P\mathcal{K}_j \in FP\mathcal{K}_A\ o \in \underline{apr}_{A,j}(\mathcal{T})\}, &(34)\\
P\underline{apr}_A(\mathcal{T}) &= \{o \in U \mid \exists P\mathcal{K}_j \in FP\mathcal{K}_A\ o \in \underline{apr}_{A,j}(\mathcal{T})\}, &(35)\\
S\overline{apr}_A(\mathcal{T}) &= \{o \in U \mid \forall P\mathcal{K}_j \in FP\mathcal{K}_A\ o \in \overline{apr}_{A,j}(\mathcal{T})\}, &(36)\\
P\overline{apr}_A(\mathcal{T}) &= \{o \in U \mid \exists P\mathcal{K}_j \in FP\mathcal{K}_A\ o \in \overline{apr}_{A,j}(\mathcal{T})\}. &(37)
\end{aligned}
$$

From these approximations, we obtain two rough sets: certain rough sets $(S\underline{apr}_A(\mathcal{T}), S\overline{apr}_A(\mathcal{T}))$[‡] and possible rough sets $(P\underline{apr}_A(\mathcal{T}), P\overline{apr}_A(\mathcal{T}))$. Namely, the rough sets are twofold under incomplete information, as is shown in [14]. Using Proposition 9, we transform the four approximations into:

$$
\begin{aligned}
S\underline{apr}_A(\mathcal{T}) &= \underline{apr}_{A,max}(\mathcal{T}), &(38)\\
P\underline{apr}_A(\mathcal{T}) &= \underline{apr}_{A,min}(\mathcal{T}), &(39)\\
S\overline{apr}_A(\mathcal{T}) &= \overline{apr}_{A,min}(\mathcal{T}), &(40)\\
P\overline{apr}_A(\mathcal{T}) &= \overline{apr}_{A,max}(\mathcal{T}), &(41)
\end{aligned}
$$

where $\underline{apr}_{A,min}(\mathcal{T})$ and $\overline{apr}_{A,min}(\mathcal{T})$ are the approximations from the minimum possible covering, and $\underline{apr}_{A,max}(\mathcal{T})$ and $\overline{apr}_{A,max}(\mathcal{T})$ are the approximations from the maximum possible covering.

Similarly to the case of handling missing values [14], the following proposition holds.

Proposition 10. $S\underline{apr}_A(\mathcal{T}) \subseteq P\underline{apr}_A(\mathcal{T}) \subseteq \mathcal{T} \subseteq S\overline{apr}_A(\mathcal{T}) \subseteq P\overline{apr}_A(\mathcal{T})$.

Proof. From formulae (34)–(37), $S\underline{apr}_A(\mathcal{T}) \subseteq P\underline{apr}_A(\mathcal{T})$ and $S\overline{apr}_A(\mathcal{T}) \subseteq P\overline{apr}_A(\mathcal{T})$. Clearly, $P\underline{apr}_A(\mathcal{T}) \subseteq \mathcal{T} \subseteq S\overline{apr}_A(\mathcal{T})$. Thus, this proposition holds.

Example 8. *We go back to Example 7. Using formulae (38) – (41), we derive*

$$
\begin{aligned}
S\underline{apr}_{a_1}(\mathcal{T}) &= \{2\},\\
P\underline{apr}_{a_1}(\mathcal{T}) &= \{2, 5\},\\
S\overline{apr}_{a_1}(\mathcal{T}) &= \{2, 5\},\\
P\overline{apr}_{a_1}(\mathcal{T}) &= \{2, 4, 5\}.
\end{aligned}
$$

‡ This expression is an interval set, as is addressed in [15]

Twofold rough sets, certain and possible rough sets, are:

$$
\begin{aligned}
(S\underline{apr}_{a_1}(\mathcal{T}), S\overline{apr}_{a_1}(\mathcal{T})) &= (\{2\}, \{2, 5\}),\\
(P\underline{apr}_{a_1}(\mathcal{T}), P\overline{apr}_{a_1}(\mathcal{T})) &= (\{2, 5\}), \{2, 4, 5\}).
\end{aligned}
$$

Under minimum possibly indiscernible class $PK(o)_{A,min}$ and maximum possibly indiscernible class $PK(o)_{A,max}$, the following formulae are obtained from formulae (38)–(41):

$$S\underline{apr}_A(\mathcal{T}) = \{o \mid PK(o)_{A,max} \subseteq \mathcal{T}\}, \quad (42)$$

$$P\underline{apr}_A(\mathcal{T}) = \{o \mid PK(o)_{A,min} \subseteq \mathcal{T}\}, \quad (43)$$

$$S\overline{apr}_A(\mathcal{T}) = \{o \mid PK(o)_{A,min} \cap \mathcal{T} \neq \emptyset\}, \quad (44)$$

$$P\overline{apr}_A(\mathcal{T}) = \{o \mid PK(o)_{A,max} \cap \mathcal{T} \neq \emptyset\}. \quad (45)$$

These formulae show that approximations obtained in this approach are equal to those derived using the minimum and the maximum possibly indiscernible classes $PK(o)_{A,min}$ and $PK(o)_{A,max}$.

Proposition 11. *There exist possible tables from which $PK(o)_{A,min}$ and $PK(o)_{A,max}$ can be derived.*

Proof. From formulae (14), (15), (19), and (21), $PK(o)_{A,min} = \{o' \in U \mid (o = o') \vee (\forall a \in A \quad \forall v \in a(o) \forall v' \in a(o') sim(v, v') \geq \delta_a)\}$ and $PK(o)_{A,max} = \{o' \in U \mid \forall a \in A \quad \exists v \in a(o) \exists v' \in a(o') sim(v, v') \geq \delta_A\}$. v and v' are a possible value in $a(o)$ and $a(o')$, respectively. Thus, this proposition holds.

Proposition 11 justifies direct derivation using minimum and maximum possible indiscernible classes from the perspective of possible world semantics.

Example 9. *From example 5, for example, $PK(5)_{a_1,min} = \{5\}$ and $PK(5)_{a_1,max} = \{1, 3, 5\}$. $PK(5)_{a_1,min}$ and $PK(5)_{a_1,max}$ can be derived from possible tables PT_3 and PT_4 in example 4, and PT_1 and PT_2, respectively.*

Lastly, we touch the case that target set $\mathcal{T}$ contains incompleteness. Let $o \in \mathcal{T}$ be specified by restriction R.

$$S\underline{apr}_A(\mathcal{T}) = \underline{apr}_{A,max}(S\mathcal{T}_R), \quad (46)$$

$$P\underline{apr}_A(\mathcal{T}) = \underline{apr}_{A,min}(P\mathcal{T}_R), \quad (47)$$

$$S\overline{apr}_A(\mathcal{T}) = \overline{apr}_{A,min}(S\mathcal{T}_R), \quad (48)$$
$$P\overline{apr}_A(\mathcal{T}) = \overline{apr}_{A,max}(P\mathcal{T}_R), \quad (49)$$

where $S\mathcal{T}_R$ and $P\mathcal{T}_R$ are sets of objects that certainly and possibly satisfy restriction R, respectively. Note that $P\underline{apr}_A(\mathcal{T}) \subseteq S\overline{apr}_A(\mathcal{T})$ does not hold when $\mathcal{T}$ contains incompleteness.

Example 10. *In incomplete information table IT of Example 4, let $\mathcal{T}$ be specified by restriction $a_2 = z$. $S\mathcal{T}_{a_2=z} = \{2, 5\}$ and $P\mathcal{T}_{a_2=z} = \{2, 4, 5\}$. The possible minimum and maximum coverings on a_1 are $P\mathcal{K}_{a_1,1}$ and $P\mathcal{K}_{a_1,16}$ in Example 6. Under formulae (46) – (49),*

$$S\underline{apr}_{a_1}(\mathcal{T}) = \{2\},$$
$$P\underline{apr}_{a_1}(\mathcal{T}) = \{2, 5\},$$
$$S\overline{apr}_{a_1}(\mathcal{T}) = \{2, 5\},$$
$$P\overline{apr}_{a_1}(\mathcal{T}) = \{2, 4, 5\}.$$

The following rules are derived from twofold rough sets that consist of the four approximations. Let $\mathcal{T}$ be specified by restriction R.

- When $o \in S\underline{apr}_A(\mathcal{T})$, rule $A = A(o) \rightarrow R$ certainly holds with accuracy 1 and support $|PK(o)_{A,min}|/|U|$. The rule is a certain and consistent one.
- When $o \in (P\underline{apr}_A(\mathcal{T}) \backslash S\underline{apr}_A(\mathcal{T}))$, rule $A = A(o) \rightarrow R$ possibly holds with accuracy 1 and support $|PK(o)_{A,max} \cap P\mathcal{T}_R|/|U|$. The rule is a possible and consistent one.
- When $o \in (S\overline{apr}_A(\mathcal{T}) \backslash S\underline{apr}_A(\mathcal{T}))$, rule $A = A(o) \rightarrow R$ certainly holds with accuracy $|PK(o)_{A,min} \cap S\mathcal{T}_R|/|PK(o)_{A,max}|$ and support $|PK(o)_{A,min} \cap S\mathcal{T}_R|/|U|$. The rule is a certain and inconsistent one.
- When $o \in (P\overline{apr}_A(\mathcal{T}) \backslash P\underline{apr}_A(\mathcal{T}) \backslash S\overline{apr}_A(\mathcal{T}))$, rule $A = A(o) \rightarrow R$ possibly holds with accuracy $|PK(o)_{A,max} \cap P\mathcal{T}_R|/|PK(o)_{A,max}|$ and support $|PK(o)_{A,max} \cap P\mathcal{T}_R|/|U|$. The rule is a possible and inconsistent one.

Example 11. *Let go back to Example 10. Object 2 is included in $S\underline{apr}_{a_1}(\mathcal{T})$. From this, rule $(a_1 = b \vee a_1 = e) \rightarrow a_2 = z$ certainly holds with accuracy 1 and support 1/5. Object 5 is included in $(P\underline{apr}_{a_1}(\mathcal{T}) \backslash S\underline{apr}_{a_1}(\mathcal{T}))$. Rule $(a_1 = c \vee a_1 = f) \rightarrow a_2 = z$ possibly holds with accuracy 1 and support 1/5. Object 5 is also included in $(S\overline{apr}_{a_1}(\mathcal{T}) \backslash S\underline{apr}_{a_1}(\mathcal{T}))$. Rule $(a_1 = c \vee a_1 = f) \rightarrow a_2 = z$ certainly holds with accuracy 1/3 and support 1/5. Object 4 is included in $(P\overline{apr}_{a_1}(\mathcal{T}) \backslash P\underline{apr}_{a_1}(\mathcal{T}) \backslash S\overline{apr}_{a_1}(\mathcal{T}))$. Rule $a_1 = d \rightarrow a_2 = z$ possibly holds with accuracy 1/2 and support 2/5.*

4 ALGORITHMS FOR CALCULATING APPROXIMATIONS AND RULES

Four kinds of rules are derived from lower and upper approximations. This means the key point of performance is in computational complexity of deriving approximations.

The algorithm deriving a set of rules in a complete information table is as follows:

Algorithm 1 to derive approximations and rules in a complete information table
Input: U, V, AT, A, R, δ_a for $a \in A$, $sim(u, v)$ for $u, v \in V(a)$
Output: $A = A(o) \rightarrow R$ for $o \in \underline{apr}_A(\mathcal{T}_R)$,
$A = A(o) \rightarrow R$ for $o \in (\underline{apr}_A(\mathcal{T}_R) \backslash \overline{apr}_A(\mathcal{T}_R))$
Begin
Step 1:
for each $a \in A$ do
Compute I_a
end
Step 2:
for each $o \in U$ Compute $K(o)_A$
Step 3: Compute $\mathcal{T}_R$ satisfying R
Step 4: Compute $\underline{apr}_A(\mathcal{T}_R), \overline{apr}_A(\mathcal{T}_R)$
Step 5: Compute $A = A(o) \rightarrow R$ for $o \in \underline{apr}_A(\mathcal{T}_R)$,
$A = A(o) \rightarrow R$ for $o \in (\underline{apr}_A(\mathcal{T}_R) \backslash \overline{apr}_A(\mathcal{T}_R))$
End

Let n and m be the number of objects in U and the number of attributes included in A, respectively. We consider the computational complexity in the worst case. For example, the maximum number of elements in the covering is n. This is the worst case. First, the computational complexity to obtain I_a is $O(n^2)$. Thus, Step 1 has $O(n^2 * m)$. The computational order to derive $K(o)_a$ from I_a is $O(n^2)$. $K(o)_A$ are derived using $K(o)_a$. Thus, Step 2 has $O(n^{2m})$. In Step 3 taget set $\mathcal{T}_R$ is specified under restriction R. Each object is checked for whether it satisfies the restriction or not. Thus, Step 3 has $O(n)$. In Step 4, each object in $K(o)_A$ is checked whether it is or not in $\mathcal{T}_R$ for all objects in U. Thus, Step 4 has $O(n^2)$. In Step 5, a rule is derived from each object in approximations obtained in Step 4. Thus, Step 5 has $O(n)$. As a whole, Step 2 is the most time consuming. Thus, Algorithm 1 has the computational complexity $O(n^{2m})$.

Next, we describe Algorithm 2 in an incomplete information table.

Algorithm 2 to calculate approximations and rules in an incomplete information table
Input: U, V, AT, A, R, δ_a for $a \in A$, $sim(u, v)$ for $u, v \in V(a)$
Output: $A = A(o) \rightarrow R$ for $o \in \underline{apr}_{A,max}(\mathcal{ST}_R)$,
$A = A(o) \rightarrow R$ for $o \in (\underline{apr}_{A,min}(\mathcal{PT}_R) \backslash \underline{apr}_{A,max}(\mathcal{ST}_R))$,
$A = A(o) \rightarrow R$ for $o \in (\overline{apr}_{A,min}(\mathcal{ST}_R) \backslash \underline{apr}_{A,max}(\mathcal{ST}_R))$,
$A = A(o) \rightarrow R$ for $o \in (\overline{apr}_{A,max}(\mathcal{PT}_R) \backslash \underline{apr}_{A,min}(\mathcal{PT}_R) \backslash \overline{apr}_{A,min}(\mathcal{ST}_R))$

Begin
Step 1:
for each $a \in A$ do
Compute $PI_{a,min}$, $PI_{a,max}$
end
Step 2:
for each $o \in U$ Compute $PK(o)_{A,min}$, $PK(o)_{A,max}$
Step 3: Compute $\mathcal{ST}_R$, $\mathcal{PT}_R$ satisfying R
Step 4: Compute $\underline{apr}_{A,max}(\mathcal{ST}_R)$, $\underline{apr}_{A,min}(\mathcal{PT}_R)$, $\overline{apr}_{A,min}(\mathcal{ST}_R)$, $\overline{apr}_{A,max}(\mathcal{PT}_R)$
Step 5: Compute
$A = A(o) \rightarrow R$ for $o \in \underline{apr}_{A,max}(\mathcal{ST}_R)$,
$A = A(o) \rightarrow R$ for $o \in (\underline{apr}_{A,min}(\mathcal{PT}_R) \backslash \underline{apr}_{A,max}(\mathcal{ST}_R))$,
$A = A(o) \rightarrow R$ for $o \in (\overline{apr}_{A,min}(\mathcal{ST}_R) \backslash \underline{apr}_{A,max}(\mathcal{ST}_R))$,
$A = A(o) \rightarrow R$ for $o \in (\overline{apr}_{A,max}(\mathcal{PT}_R) \backslash \underline{apr}_{A,min}(\mathcal{PT}_R) \backslash \overline{apr}_{A,min}(\mathcal{ST}_R))$
End

In Step 1, $PI_{a,min}$ and $PI_{a,max}$ are derived using formulae (14) and (15). The computational complexity is $O(n^2)$. In Step 2, $PK(o)_{a,min}$ and $PK(o)_{a,max}$ are calculated using formulae (19) and (21). $PK(o)_{A,min}$ and $PK(o)_{A,max}$ are derived from formulae (23) and (24). Thus, the computational complexity is $O(n^{2m})$. In Step 3, $\mathcal{ST}_R$ and $\mathcal{PT}_R$ are sets of objects that certainly and possibly satisfy restriction R, respectively. Thus, the computational complexity is $O(n)$. In Step 4, $\underline{apr}_{A,max}(\mathcal{ST}_R)$, $\underline{apr}_{A,min}(\mathcal{PT}_R)$, $\overline{apr}_{A,min}(\mathcal{ST}_R)$, and $\overline{apr}_{A,max}(\mathcal{PT}_R)$ are derived using formulae (42)–(45). Thus, the computational complexity is $O(n^2)$. In Step 5, a rule is derived from each object in four approximations obtained in Step 4. Thus, Step 5 has $O(n)$. As a whole, Step 2 is the most time consuming like Algorithm 1. Thus, Algorithm 2 has the computational complexity $O(n^{2m})$, which is the same as Algorithm 1. Namely, rough sets in incomplete information tables have the same computational complexity for the number of objects as ones in complete information tables.

Cao et al. describe an algorithm of calculating approximations for the case of dealing with missing values under possible equivalence classes [27]. The computational complexity of our approach is the same as that of Cao et al. In addition, Cao et al. show that parallel computing can be used to overcome the computational complexity of Step 2. The Introduction of parallel computing is expected to play an important role in improving efficiency.

5 CONCLUSIONS

We have described rough sets and rule induction based on coverings in information tables with similarity of values. The similarity degree of values characterizing objects is reflexive, asymmetric, and not transitive. In a complete information table, the covering on a set of attributes is unique. Two kinds of rules are induced from lower and upper approximations that correspond to the inclusion and intersection of granules to a target set.

In an incomplete information table, lots of coverings, called possible coverings, are derived on a set of attributes. The family of possible coverings is a lattice with the minimum and maximum elements. This means that this family does not newly cause difficulty due to computational complexity for obtaining rough sets. This is also true for the family of possible maximum description, but not for the family of possible minimum description and the family of possible sets of close friends.

Four approximations are derived using only the minimum and maximum possible coverings that are derived from the minimum and maximum possible indiscernibility relations, which are equal to the intersection and the union of indiscernibility relations from possible tables, respectively. The four approximations produce the twofold rough sets: certain rough sets and possible rough sets. These approximations are equal to those derived using the minimum and the maximum possibly indiscernible classes. These classes can be also obtained from possible tables. This justifies the approach.

From the twofold rough sets, we obtain four kinds of rules with accuracy and support. The computational complexity is the same order of the number of objects as in a complete information table.

Further research is to first conduct experiments using actual data. This approach is then extended to handle more actual data in the following directions. One is to develop an approach with an incomplete information table with ordered domains. Ordered values often appear in the real world. For example, the age of a person, the price of an item, etc. [28]. The other is to handle incomplete information expressed in a possibility distribution. As shown in Zadeh [29], the possibility distribution is suitable for expressing vague values in the real world.

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