J. of Mult.-Valued Logic & Soft Computing, Vol. 43, pp. 343–354
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Intuitionistic Fuzzy Interpretation of Quantum Logic Axioms

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Accepted: April 18, 2024.

In 1936 G. Birkhoff and J. von Neumann introduced the concept of a Quantum Logic. In 2007 M. Pavičić and N. Megill introduced one of the axioms of this logic. The Intuitionistic Fuzzy Logic (IFL) is an extension of L. Zadeh's fuzzy logic. The IFL is used as a tool for the interpretation of quantum logic axioms of Pavičić and Megill. Initially, we use the most popular forms of the IFL-operations implication and negation, and the conjunction and disjunction generated by them. Subsequently, we provide illustrations with use of other IFL-operations for quantum logic axioms interpretation. The advantages of these interpretations and potential directions for further research are discussed, e.g., the possibility of extending the quantum logic axioms in the directions of temporal and modal logics, the possibility for using other types of logical operations, and others.

Keywords: Axiom, Intuitionistic fuzzy logic, Quantum logic

AMS: 03E72;03G12

1 INTRODUCTION

The concept of Quantum Logic (QL) was originally introduced in [8] by G. Birkhoff and J. von Neumann as *an idea of a "logic of quantum mechanics"* *or quantum logic* [7]. In the next years, some important research, related to QL were published (see, e.g., [1, 2, 6, 9–17]). It is important to mention that the first attempt for an intuitionistic fuzzy interpretation of Quantum Logic axioms is published in [16].

In [15] the following axioms of QL are given.

QLA1: $A \equiv A$
QLA2: $(A \equiv B) \Rightarrow_0 (B \equiv C \Rightarrow_0 A \equiv C)$
QLA3: $(A \equiv B) \Rightarrow_0 (\neg A \equiv \neg B)$
QLA4: $(A \equiv B) \Rightarrow_0 (A \wedge C \equiv B \wedge C)$
QLA5: $(A \wedge B) \equiv (B \wedge A)$
QLA6: $(A \wedge (B \wedge C)) \equiv ((A \wedge B) \wedge C)$
QLA7: $(A \wedge (A \vee B)) \equiv A$
QLA8: $(\neg A \wedge A) \equiv ((\neg A \wedge A) \wedge B)$
QLA9: $A \equiv \neg\neg A$
QLA10: $\neg(A \vee B) \equiv (\neg A \wedge \neg B)$
QLA11: $(A \vee (\neg A \wedge (A \vee B))) \equiv (A \vee B)$
QLA12: $(A \equiv B) \equiv (B \equiv A)$
QLA13: $A \equiv (B \Rightarrow_0 (A \Rightarrow_0 B))$
QLA14: $(A \Rightarrow_0 B) \Rightarrow_3 (A \Rightarrow_3 (A \Rightarrow_3 B))$
QLA15: $(A \Rightarrow_3 B) \Rightarrow_0 (A \Rightarrow_0 B)$

where the symbols $\equiv, \Rightarrow_0, \Rightarrow_3, \wedge, \vee, \neg$ denote respectively operations "equivalence", "implication" (in [15] two different types of implication are used and numbered as it is written above), "conjunction", "disjunction" and "negation".

The idea for intuitionistic fuzziness was introduced in 1983 (see [4]) as an extension of L. Zadeh's fuzziness [18]. In the intuitionistic fuzziness, each object is evaluated not only with a degree of membership, validity, etc. but also with a degree of non-membership, non-validity, etc. The sum of both degrees is smaller or equal to 1. So, a third-degree – that of uncertainty is generated as a complement of the sum of both degrees to 1. Therefore, the intuitionistic fuzzy evaluations give more detailed information for the evaluated objects. Currently, the apparatus of the intuitionistic fuzzy sets and logics (propositional, predicate, modal, temporal) have a lot of applications, for example, in Artificial Intelligence, different sciences such as physics, biology, medicine, ecology, economy, etc.

In Section 3, Intuitionistic Fuzzy Interpretation (IFI) of the QL axioms will be given. It is based on the concept of an Intuitionistic Fuzzy Logic (IFL, see [4]) in its part Intuitionistic Fuzzy Propositional Calculus. Because the authors of [15] use two different implications in QL axioms, we also use two different intuitionistic fuzzy implications, but in addition, we show that the forms of the implications can be different and the results will also be different (cf. Theorems 1 and 3).

In Section 2 short remarks on the used concepts are given. Some ideas for modifications and extensions of the QL-axioms will be discussed in the Conclusion, because the introduced here approach to QL axioms give essentially larger possibilities for development of the QL and, potentially its applications.

2 PRELIMINARIES

In the present paper, we use some concepts from the apparatus of Intuitionistic Fuzzy Propositional Calculus.

Following [5], the ordered pair $\langle a, b\rangle$, where $a, b, a + b \in [0, 1]$ is called an Intuitionistic Fuzzy Pair (IFP). In it, a and b represent the degrees of validity (membership, etc.) and of non-validity (non-membership, etc.), respectively. Therefore, in this case, the number $1 - a - b$ determine the degree of uncertainty (indeterminacy).

Obviously, each fuzzy evaluation (a) can be represented by an IFP in the form $\langle a, 1 - a\rangle$. For it $1 - a - (1 - a) = 0$, i.e., there is no uncertainty. So, it is clear, that the intuitionistic fuzzy evaluations give more detailed information than the fuzzy evaluations.

The IFP $\langle a, b\rangle$ is:
a tautology if and only if (iff) $a = 1$ and $b = 0$;
an intuitionistic fuzzy tautology (IFT) iff $a \geq b$.

Therefore, if the IFP is a tautology, then it is an IFT, but the opposite is not true.

We must mention that if some logical object (variable, term, sentence, formula, etc.) has truth-value $\langle a, b\rangle$, where a and b are its intuitionistic fuzzy degrees and if we prove that $a = 1, b = 0$ or that $a \geq b$ for each possible interpretation of the object, then this object is a tautology or an IFT, respectively, in the sense of the intuitionistic fuzziness point of view. In Section 3, we illustrate both types of tautologies.

Let everywhere below, the intuituinistic fuzzy truth values of variables A, B, C be $\langle a, b\rangle, \langle c, d\rangle, \langle e, f\rangle$, respectively.

For two IFPs A and B, different relations and operations are introduced in [4, 5], but we will use only the following two of these relations:

$$A \leq B \text{ iff } a \leq c \text{ and } b \geq d,$$

$$A = B \text{ iff } a = c \text{ and } b = d.$$

and the following operations:

$$A\&B = \langle \min(a, c), \max(b, d)\rangle,$$
$$A \vee B = \langle \max(a, c)), \min(b, d)\rangle,$$

$$A \to B = \langle \max(b, c)), \min(a, d)\rangle,$$
$$\neg A = \langle b, a\rangle.$$

3 MAIN RESULTS

First, following [15], we give an IFI of the operations, used in the QL-axioms:

$$A \Rightarrow_0 B = \neg A \vee B = \langle \max(b, c), \min(a, d)\rangle,$$

$$\begin{aligned} A \Rightarrow_3 B &= (\neg A \& B) \vee (\neg A \& \neg B) \vee (A \& (\neg A \vee B)) \\ &= \langle \max(\min(b, c), \min(b, d), \min(a, \max(b, c))), \\ &\quad \min(\max(a, c), \max(a, d), \max(b, \min(a, d)))\rangle. \end{aligned}$$

$$\begin{aligned} A \equiv B &= (A \& B) \vee (\neg A \& \neg B) \\ &= \langle \max(\min(a, c), \min(b, d)), \min(\max(a, c), \max(b, d))\rangle. \end{aligned}$$

In practice, implication $\Rightarrow_0$ coincides with the fourth intuitionistic fuzzy implication (see, e.g., [4])

$$A \Rightarrow_4 B = \langle \max(b, c), \min(a, d)\rangle,$$

while implication $\Rightarrow_3$ (from [15]) is its derivative.

Second, we prove the following theorem.

Theorem 1. *Axioms QLA1 – QLA12, QLA14 and QLA15 are all IFTs, but none is a tautology.*

Proof. Below, we show why QLA13 is not an IFT and the proof why the remaining ones are IFTs is similar, and this is illustrated by the proof that QLA15 is an IFT.

$$\begin{aligned} A \equiv (B \Rightarrow_0 (A \Rightarrow_0 B)) &= \langle a, b\rangle \equiv (\langle c, d\rangle \Rightarrow_0 (\langle a, b\rangle \Rightarrow_0 \langle c, d\rangle)) \\ &= \langle a, b\rangle \equiv (\langle c, d\rangle \Rightarrow_0 \langle \max(b, c), \min(a, d)\rangle) \\ &= \langle a, b\rangle \equiv \langle \max(b, c, d), \min(a, c, d)\rangle \\ &= \langle \max(\min(a, \max(b, c, d)), \min(b, \min(a, c, d))), \\ &\quad \min(\max(a, \max(b, c, d)), \max(b, \min(a, c, d)))\rangle. \end{aligned}$$

Let

$$\begin{aligned} X &= \max(\min(a, \max(b, c, d)), \min(b, \min(a, c, d))) \\ &\quad - \min(\max(a, \max(b, c, d)), \max(b, \min(a, c, d))). \end{aligned}$$

If $a \geq \max(b, c, d)$, then

$$\begin{aligned} X &= \max(\max(b, c, d), \min(b, \min(a, c, d))) - \min(a, \max(b, \min(a, c, d))) \\ &= \max(b, c, d) - \min(a, \max(b, \min(a, c, d))). \end{aligned}$$

If $b \geq \min(a, c, d)$, then

$$X = \max(b, c, d) - \min(a, b) \geq b - \min(a, b) \geq 0.$$

If $b < \min(a, c, d)$, then

$$X = \max(b, c, d) - \min(a, \min(a, c, d)) = \max(b, c, d) - \min(a, c, d)) \geq 0.$$

If $a < \max(b, c, d)$, then

$$\begin{aligned} X &= \max(a, \min(a, b, c, d)) - \min(\max(b, c, d), \max(b, \min(a, c, d))) \\ &= a - \min(\max(b, c, d), \max(b, \min(a, c, d))). \end{aligned}$$

Now, we see that when $a = 1, b = 0$,

$$X = 1 - \max(c, d) \geq 0,$$

while, when $a = 0, b = 1$,

$$X = 0 - \min(1, 1) = -1.$$

Therefore, QLA13 is not an IFT and hence, it is not a tautology.

It is important to note that QLA13 is not a tautology in the classical sense, too.

For QLA15 we obtain:

$$\begin{aligned}
&(A \Rightarrow_3 B) \Rightarrow_0 (A \Rightarrow_0 B)\\
&= \langle \max(\min(b, c), \min(b, d), \min(a, \max(b, c))),\\
&\quad \min(\max(a, c), \max(a, d), \max(b, \min(a, d)))\rangle \Rightarrow_0 \langle \max(b, c), \min(a, d)\rangle\\
&= \langle \max(\min(\max(a, c), \max(a, d), \max(b, \min(a, d))), \max(b, c)),\\
&\quad \min(\max(\min(b, c), \min(b, d), \min(a, \max(b, c))), \min(a, d))\rangle.
\end{aligned}$$

Let

$$\begin{aligned}
X &= \max(\min(\max(a, c), \max(a, d), \max(b, \min(a, d))), \max(b, c))\\
&\quad - \min(\max(\min(b, c), \min(b, d), \min(a, \max(b, c))), \min(a, d)).
\end{aligned}$$

If $b \geq \min(a, d)$, then, having in mind that

$$\max(b, c) \geq b \geq \min(\max(a, c), \max(a, d), b)$$

we obtain:

$$\begin{aligned}
X &= \max(\min(\max(a, c), \max(a, d), b), \max(b, c))\\
&\quad - \min(\max(\min(b, c), \min(b, d), \min(a, \max(b, c))), \min(a, d))\\
&\geq \max(b, c) - \min(\max(\min(b, c), \min(b, d), \min(a, \max(b, c))), b)\\
&\geq \max(b, c) - b \geq 0.
\end{aligned}$$

If $b < \min(a, d)$, then $\max(a, c) \geq a \geq \min(a, d)$ and

$$\begin{aligned}
X &= \max(\min(a, d), \max(b, c)) - \min(\max(\min(b, c), \min(b, d),\\
&\qquad \min(a, \max(b, c))), \min(a, d)).\\
&\geq \min(a, d) - \min(\max(\min(b, c), \min(b, d), \min(a, \max(b, c))), \min(a, d)) \geq 0.
\end{aligned}$$

Therefore, QLA15 is an IFT. But, for example, if $a = b = c = d = 0.5$, then we calculate directly that

$$(A \Rightarrow_3 B) \Rightarrow_0 (A \Rightarrow_0 B) = \langle 0.5, 0.5\rangle,$$

i.e., in this case QLA15 is an IFT only, and is not a tautology. □

Third, we can introduce the following different (weak) form of QLA13:

QLA13*: $A \Rightarrow_0 (B \Rightarrow_0 (A \Rightarrow_0 B))$.

and for it we can prove the following theorem.

Theorem 2. *Axiom QLA13* is an IFT, but it is not a tautology.*

Fourth, we can substitute negation $\neg$ (that in IFL is marked as $\neg_1$, see [4]) with another one, e.g.

$$\neg_2 = \langle \overline{\text{sg}}(a), \text{sg}(a) \rangle,$$

where

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \qquad \overline{\text{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

Therefore, we obtain a new intuitionistic fuzzy implication (in IFL it is marked as $\rightarrow_2$, see [4]):

$$A \rightarrow_2 B = \langle \overline{\text{sg}}(a - c), d\,\text{sg}(a - c) \rangle$$

that can be used on the place of implication $\Rightarrow_0$ in the above QL-axioms. Using negation $\neg_2$ in the formula for above implication $\Rightarrow_3$ and keeping the above forms for operations $\vee$ and &, we obtain its new form as

$$\begin{aligned} \langle a, b\rangle \Rightarrow_3 \langle c, d\rangle = \langle &\max(\min(\overline{\text{sg}}(a), c), \min(\overline{\text{sg}}(a), \overline{\text{sg}}(c)), \min(\max(\overline{\text{sg}}(a), c), a)), \\ &\min(\max(\text{sg}(a), \text{sg}(c)), \max(\text{sg}(a), d), \max(b, \min(\text{sg}(a), d)))\rangle. \end{aligned}$$

In this case, we can prove in a similar manner as Theorem 1 the following theorem.

Theorem 3. *Axioms QLA1 – QLA12 and QLA15 are all IFTs, but none of them is a tautology.*

On the other side, if we define operations $\vee$ and & using negation $\neg_2$ and implication $\rightarrow_2$, by

$$\begin{aligned} \langle a, b\rangle \vee \langle c, d\rangle &= \neg\langle a, b\rangle \rightarrow_2 \langle c, d\rangle \\ &= \langle \overline{\text{sg}}(a), \text{sg}(a)\rangle \rightarrow_2 \langle c, d\rangle \\ &= \langle \overline{\text{sg}}(\overline{\text{sg}}(a) - c), d\,\text{sg}(\overline{\text{sg}}(a) - c)\rangle, \end{aligned}$$

$$
\begin{aligned}
\langle a,b\rangle \& \langle c,d\rangle &= \neg(\langle a,b\rangle \rightarrow_2 \neg\langle c,d\rangle)\\
&= \neg(\langle a,b\rangle \rightarrow_2 \langle \overline{\mathrm{sg}}(c), \mathrm{sg}(c)\rangle)\\
&= \neg\langle \overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(c)), \mathrm{sg}(c)\mathrm{sg}(a-\overline{\mathrm{sg}}(c))\rangle\\
&= \langle \mathrm{sg}(a-\overline{\mathrm{sg}}(c)), \overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(c))\rangle,
\end{aligned}
$$

because for each $x \in [-1, 1]$:

$$
\begin{aligned}
\mathrm{sg}(\mathrm{sg}(x)) &= \mathrm{sg}(x),\\
\mathrm{sg}(\overline{\mathrm{sg}}(x)) &= \overline{\mathrm{sg}}(x),\\
\overline{\mathrm{sg}}(\mathrm{sg}(x)) &= \overline{\mathrm{sg}}(x),\\
\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(x)) &= \mathrm{sg}(x),
\end{aligned}
$$

then for the operation $\equiv$ we obtain

$$
\begin{aligned}
A \equiv B &= (A\&B) \vee (\neg A \& \neg B)\\
&= \langle \mathrm{sg}(a-\overline{\mathrm{sg}}(c)), \overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(c))\rangle \vee (\langle \overline{\mathrm{sg}}(a), \mathrm{sg}(a)\rangle \& \langle \overline{\mathrm{sg}}(c), \mathrm{sg}(c)\rangle)\\
&= \langle \mathrm{sg}(a-\overline{\mathrm{sg}}(c)), \overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(c))\rangle \vee \langle \mathrm{sg}(\overline{\mathrm{sg}}(a)-\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(c))), \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)-\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(c)))\rangle\\
&= \langle \mathrm{sg}(a-\overline{\mathrm{sg}}(c)), \overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(c))\rangle \vee \langle \mathrm{sg}(\overline{\mathrm{sg}}(a)-\mathrm{sg}(c)), \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)-\mathrm{sg}(c))\rangle\\
&= \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(\mathrm{sg}(a-\overline{\mathrm{sg}}(c))) - \mathrm{sg}(\overline{\mathrm{sg}}(a)-\mathrm{sg}(c))),\\
&\qquad \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)-\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(c)))\mathrm{sg}(\overline{\mathrm{sg}}(\mathrm{sg}(a-\overline{\mathrm{sg}}(c))) - \mathrm{sg}(\overline{\mathrm{sg}}(a)-\mathrm{sg}(c)))\rangle\\
&= \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(c)) - \mathrm{sg}(\overline{\mathrm{sg}}(a)-\mathrm{sg}(c))),\\
&\qquad \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)-\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(c)))\mathrm{sg}(\overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(c)) - \mathrm{sg}(\overline{\mathrm{sg}}(a)-\mathrm{sg}(c)))\rangle.
\end{aligned}
$$

For the last forms of the operations, the validity of the next theorem follows.

Theorem 4. *Axioms QLA1, QLA2, ..., QLA9, QLA12 and QLA15 are tautologies and therefore IFTs, too.*

Really, for example, for the IFI of QLA1 we obtain

$$
\begin{aligned}
A \equiv A = \langle &\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(a)) - \mathrm{sg}(\overline{\mathrm{sg}}(a)-\mathrm{sg}(a))),\\
&\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)-\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)))\mathrm{sg}(\overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(a)) - \mathrm{sg}(\overline{\mathrm{sg}}(a)-\mathrm{sg}(a)))\rangle.
\end{aligned}
$$

Let

$$X = \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(a)) - \mathrm{sg}(\overline{\mathrm{sg}}(a)-\mathrm{sg}(a))).$$

Then

$$X = \begin{cases} \text{if } a = 0 : \overline{\text{sg}}(\overline{\text{sg}}(-1) - \text{sg}(1)) = \overline{\text{sg}}(0) = 1, \\ \text{if } 0 < a < 1 : \overline{\text{sg}}(\overline{\text{sg}}(a) - \text{sg}(-1)) = \overline{\text{sg}}(0) = 1, \\ \text{if } a = 1 : \overline{\text{sg}}(\overline{\text{sg}}(1) - \text{sg}(-1)) = \overline{\text{sg}}(0) = 1. \end{cases}$$

Let

$$Y = \overline{\text{sg}}(\overline{\text{sg}}(a) - \overline{\text{sg}}(\overline{\text{sg}}(a)))\text{sg}(\overline{\text{sg}}(a - \overline{\text{sg}}(a)) - \text{sg}(\overline{\text{sg}}(a) - \text{sg}(a))).$$

Then

$$Y = \begin{cases} \text{if } a = 0 : \overline{\text{sg}}(1)\text{sg}(\overline{\text{sg}}(-1) - \text{sg}(1)) = 0, \\ \text{if } 0 < a < 1 : \overline{\text{sg}}(-1)\text{sg}(\overline{\text{sg}}(a) - \text{sg}(-1)) = 0, \\ \text{if } a = 1 : \overline{\text{sg}}(-1)\text{sg}(\overline{\text{sg}}(a) - \text{sg}(-1)) = 0. \end{cases}$$

Therefore, in all cases the truth value of $A \equiv A$ is $\langle 1, 0 \rangle$, i.e., QLA1 is a tautology (and therefore, it is an IFT, too).

Fifth, keeping negation $\neg_1$ and representing implication $\Rightarrow_3$ by implication

$$\langle a, b \rangle \rightarrow_{72} \langle c, d \rangle = \langle \max(b, c), \min(a, 1 - c) \rangle$$

(see [4]), we can give a third interpretation of the QL-axioms. For this aim, we must check the validity of axioms QLA14 and QLA15, as follows, having in mind that $a \leq 1 - b$:

$$\begin{aligned} &(A \Rightarrow_0 B) \Rightarrow_3 (A \Rightarrow_3 (A \Rightarrow_3 B)) \\ &= \langle \max(b, c), \min(a, d) \rangle \rightarrow_{72} (\langle a, b \rangle \rightarrow_{18} \langle \max(b, c), \min(a, 1 - c) \rangle) \\ &= \langle \max(b, c), \min(a, d) \rangle \rightarrow_{72} \langle \max(b, c), \min(a, 1 - \max(b, c)) \\ &= \langle \max(b, c), \min(a, d) \rangle \rightarrow_{72} \langle \max(b, c), \min(a, 1 - b, 1 - c)) \\ &= \langle \max(b, c), \min(a, d) \rangle \rightarrow_{72} \langle \max(b, c), \min(a, 1 - c)) \\ &= \langle \max(\min(a, d), b, c), \min(\max(b, c), 1 - \max(b, c)) \rangle. \end{aligned}$$

From inequalities

$$\max(\min(a, d), b, c) \geq \max(b, c) \geq \min(\max(b, c), 1 - \max(b, c))$$

it follows the validity of axiom QLA14. Axiom QLA15 is checked in the same way.

4 CONCLUSION

Now we will discuss the advantages of applying the IFI to QL-axioms.

Firstly, IFI provides the potential for a significantly more detailed interpretation of the uncertainty that exists in quantum mechanics, having in mind that in IFL, the uncertainty is represented in an explicit form: for the IFP $\langle a, b\rangle$, it is $1 - a - b$.

Secondly, as we saw above, using the apparatus of IFL, the conjunctions, disjunctions, negations and implications in the QL-axioms can have other forms. For example, we can use the operations

$$\begin{aligned} x \vee y &= \langle a + c - a.c, b.d\rangle, \\ x \& y &= \langle a.c, b + d - b.d\rangle, \\ x \rightarrow y &= \langle b + c - b.c, a.d\rangle \end{aligned}$$

used in [16] – probably the first paper in which the relation between the intuitionistic fuzziness and quantum computing is discussed.

So, it is possible to explore more effective operations and introduce new and intriguing axioms. For example, one of the basic properties of QL that the distributive law is not valid holds, e.g. for the conjunction and disjunction

$$\begin{aligned} \langle a, b\rangle \vee_2 \langle c, d\rangle &= \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a) - c), d.\mathrm{sg}(\overline{\mathrm{sg}}(a) - c)\rangle, \\ \langle a, b\rangle \&_2 \langle c, d\rangle &= \langle \overline{\mathrm{sg}}(a - \overline{\mathrm{sg}}(c)), \mathrm{sg}(a - \overline{\mathrm{sg}}(c))\rangle, \end{aligned}$$

generated by the second implication $\rightarrow_2$. Indeed,

$$\begin{aligned} &\langle a, b\rangle \&_2(\langle c, d\rangle \vee_2 \langle e, f\rangle) \\ &= \langle \overline{\mathrm{sg}}(a - \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(c) - e))), \mathrm{sg}(a - \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(c) - e)))\rangle \\ &\neq \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a - \overline{\mathrm{sg}}(c)) - \overline{\mathrm{sg}}(a - \overline{\mathrm{sg}}(e)))), \mathrm{sg}(\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a - \overline{\mathrm{sg}}(c)) - \overline{\mathrm{sg}}(a - \overline{\mathrm{sg}}(e))))\rangle \\ &= (\langle a, b\rangle \vee_2 \langle c, d\rangle)\&_2(\langle a, b\rangle \vee_2 \langle e, f\rangle). \end{aligned}$$

Thirdly, IFI can include not only the standard modal operators, but also their extensions that exist only in IFL (see [4]). For example, in [17], the two modal operators $\Box$ and $\Diamond$ have very trivial (crisp) interpretations, while in the intuitionistic fuzzy case, they have fuzzy forms. Moreover, the extended modal operators of the IFL have intuitionistic fuzzy forms.

We will give the following example. Right at the beginning (in 1983) of the first research on IFS (see [3]) the two basic modal operators $\Box$ and $\Diamond$

obtained their IFIs as follows:

$$\Box A = \langle a, 1-a \rangle,$$

$$\Diamond A = \langle 1-b, b \rangle.$$

Therefore, if A has a fuzzy evaluation, i.e. $A = \langle a, 1-a \rangle$, then

$$\Box A = A = \Diamond A,$$

while when A is a proper IFP, i.e., $a + b < 1$, then

$$\Box A < A < \Diamond A.$$

Therefore, in the fuzzy case, both operators lose their sense, while in the intuitionistic fuzzy case they are essentially different objects.

Fourth, IFI can be extended to temporal intuitionistic fuzzy forms, based on the ideas of temporal IFL (see [4]) in which Temporal IFPs (TIFPs) can be employed. They take the form $\langle a(t), b(t) \rangle$, where for each fixed temporal scale T and for each $t \in T$, $a(t), b(t), a(t) + b(t) \in [0, 1]$. In this case, these TIFPs will correspond to the evaluations of the logical conditions existing for the dynamical processes, that is the standard case in the quantum case.

Fifth, in the future, some of the results from papers such as [2,6,9,11,12], will be subjects of intuitionistic fuzzy interpretations.

Finally, we mention that here, following R. Reiser, A. Lemke, A. Avila, J. Vieira, M. Pilla and A. Du Bois's paper [16], a partial answer of the Open problem from [3] is given.

ACKNOWLEDGEMENTS

This research was funded by the Bulgarian National Science Fund, grant number KP-06-N22-1/2018 "Theoretical research and applications of Inter-Criteria Analysis".

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