

A Property of the Intuitionistic Fuzzy Modal Operator $X_{B,C,D,F,G}$

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Received: September 15, 2024. Accepted: May 4, 2025.

In the theory of the Intuitionistic Fuzzy Sets (IFS) there exist operators of modal type, of level type, and of topological type. The modal operators themselves are from three different types. Initially, an intuitionistic fuzzy analogues of the classical modal operators “necessity” and “possibility” were introduced. Following that, they were generalized to new modal-type operators specific to IFSs. In the present paper is considered the most general (at least until this moment) of these operators. It is shown that with appropriate choice of its parameters, this operator can transform the intuitionistic fuzzy values of an arbitrary IFS to the intuitionistic fuzzy values of another arbitrarily chosen IFS.

Keywords: Intuitionistic fuzzy operator, intuitionistic fuzzy set

AMS Classification: 03E72

1 INTRODUCTION

Since the definition of intuitionistic fuzzy sets (IFSs) in 1983, many operations, relations and operators have been defined on them (see [5, 8]). Some specifically intensive research was dedicated to the operators of modal type (see [12, 15, 16, 22, 27, 28, 30, 31]) of level type (see [14, 24, 26, 33]), as well as some combined versions as modal-level operators (see [10]).

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The first modal operators $\Box$ (*Necessity*) and $\Diamond$ (*Possibility*) were described in the very first paper on IFSs from 1983 [1], and a few years later their generalizations D and F [2], H , J and H^* , J^* (or $\overline{H}$, $\overline{J}$) [3] were introduced. These operators have no analogues in modal logic (see, e.g., [17, 19, 20, 23]). They are specific to IFS, and as it was shown in [1] operators $\Box$ and $\Diamond$ defined over the standard fuzzy sets lose their meaning.

The earliest version of the operator X , which generalizes all operators $\Box$, $\Diamond$, D , F , G , H , H^* , J , J^* was proposed in 1993 [4] and later developed in 2007 in [6], where the operator's parameters a, b, c, d, e, f were set to be numerical variables from the $[0, 1]$ interval. In paper [11] from 2025 this operator was generalized by substituting its numerical parameters with whole intuitionistic fuzzy sets (which results in the reduction of the number of parameters from six to five due to the presence of two degrees in an IFS). An earlier attempt for generalization was done in 2012 in [7], but the results in it were only partial since the number of IFS parameters considered back then were only three.

While the original operator is able to transform one intuitionistic fuzzy pair (see Section 3) to another, the newly proposed operator transforms one stack of intuitionistic fuzzy pairs to another stack, and in particular, one IFS to another IFS.

In [11] this operator was subject to a new generalization. This article examines an important property of it: to transform one IFS into another.

2 PRELIMINARIES

When we have some fixed set (universe) E , over it we can construct the set

$$A^* = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in E \},$$

where $A \subseteq E$, functions $\mu_A : E \rightarrow [0, 1]$ and $\nu_A : E \rightarrow [0, 1]$ represent the *degree of membership* (validity, etc.) and *non-membership* (non-validity, etc.) of element $x \in E$ to A , and for each $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

This set was called an IFS and below, we will write A instead of A^* , for brevity.

The IFSs have a lot of geometrical interpretations. One of them – probably the most popular and informative – is shown on Fig. 1. It represents how an IFS-element x defined over the universe set E is represented in the intuitionistic fuzzy interpretation triangle.

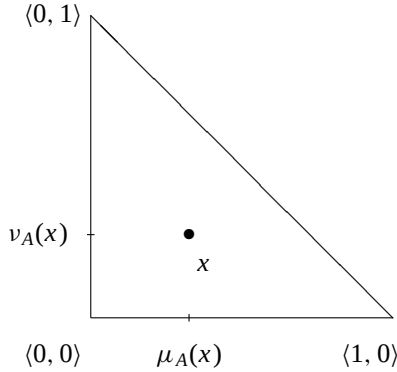


FIGURE 1
A geometrical interpretation of an element of an IFS

In [11] was introduced the operator

$$X_{B,C,D,F,G}(A) = \{ \langle x, \mu_B(x)\mu_A(x) + \mu_C(x)(1 - \mu_A(x) - \mu_D(x)\nu_A(x)), \\ \nu_F(x)\nu_A(x) + \nu_C(x)(1 - \nu_A(x) - \nu_G(x)\mu_A(x)) \rangle | x \in E \}$$

defined over the IFS A , where B, C, D, F, G are fixed IFSs, such that for each $x \in E$

$$0 \leq \mu_B(x) + \nu_C(x) - \nu_C(x)\nu_G(x) \leq 1, \quad (1)$$

$$0 \leq \mu_C(x) + \nu_F(x) - \mu_C(x)\mu_D(x) \leq 1. \quad (2)$$

In [11] it was proved that the definition of the operator $X_{B,C,D,F,G}$ is correct, that it can represent each of the extended intuitionistic fuzzy modal operators described in [8], and some of its properties are studied. Therefore, the new operator is from modal type and it is the fourth in order extension of the intuitionistic fuzzy modal operators that corresponds to the modal operators in logics (see, e.g., [17, 19, 20, 23]). Indeed, as it is shown in [8], the intuitionistic fuzzy form of the modal operators $\Box$ and $\Diamond$ is extended by the operator D_α , so that for each IFS A :

$$\Box A = D_0(A), \quad \Diamond A = D_1(A).$$

The operator D_α is extended by the operator $F_{\alpha,\beta}$ so that

$$D_\alpha(A) = F_{\alpha,1-\alpha}(A),$$

where $\alpha \in [0, 1]$.

The operator $F_{\alpha,\beta}$ is extended by the operator $X_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}$ so that

$$F_{\alpha,\beta}(A) = X_{1,\alpha,1,1,\beta,1}(A),$$

where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \in [0, 1]$.

Finally, the operator $X_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}$ is extended by the operator $X_{B,C,D,F,G}$ so that

$$X_{1,\alpha,1,1,\beta,1}(A) = X_{B,C,D,F,G}(A),$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0, 1]$,

$$\alpha + \varepsilon - \varepsilon\zeta \leq 1,$$

$$\beta + \delta - \beta\gamma \leq 1,$$

$$\beta + \varepsilon \leq 1$$

and

$$B = \{\langle x, 1, 0 \rangle | x \in E\},$$

$$C = \{\langle x, \alpha, \beta \rangle | x \in E\} \text{ for } \alpha + \beta \leq 1,$$

$$D = \{\langle x, 1, 0 \rangle | x \in E\},$$

$$F = \{\langle x, 0, 1 \rangle | x \in E\},$$

$$G = \{\langle x, 0, 1 \rangle | x \in E\}.$$

3 MAIN RESULTS

For each IFS

$$A = \{\langle x, \mu_A(x), \nu_A(x) \rangle | x \in E\},$$

let us define the intuitionistic fuzzy pair (IFP, see [13])

$$A(x) = \langle x, \mu_A(x), \nu_A(x) \rangle.$$

Let us have the IFSs A, B, C, D, F, G over a given universe E , defined so that for each $x \in E$ (1) and (2) are fulfilled.

Now, following [11], let us define the IFP for which (1) and (2) are valid:

$$X_{B,C,D,F,G}(A(x)) = \langle x, \mu_B(x)\mu_A(x) + \mu_C(x)(1 - \mu_A(x) - \mu_D(x)v_A(x)), \\ v_F(x)v_A(x) + v_C(x)(1 - v_A(x) - v_G(x)\mu_A(x)) \rangle.$$

Here, we prove the following

Theorem 1. *For every two IFSS A and Z there exist IFSS B, C, D, F, G such that (1) and (2) are fulfilled and*

$$Z = X_{B,C,D,F,G}(A). \quad (3)$$

Proof. Obviously, each IFS A over universe E can be represented as

$$A = \bigcup_{x \in E} \{\langle x, \mu_A(x), v_A(x) \rangle\}.$$

Therefore, to prove the theorem, for each $x \in E$, having $A(x)$ and $Z(x)$, we must construct the objects $B(x), C(x), D(x), F(x), G(x)$ by analogy with [9] for each IFP such that

$$Z(x) = X_{B(x),C(x),D(x),F(x),G(x)}(A(x)),$$

i.e.,

$$\mu_Z(x) = \mu_B(x)\mu_A(x) + \mu_C(x)(1 - \mu_A(x) - \mu_D(x)v_A(x)), \\ v_Z(x) = v_F(x)v_A(x) + v_C(x)(1 - v_A(x) - v_G(x)\mu_A(x)).$$

We must discuss the following possible disjoint cases:

x	μ	v	Case No.
x_1	$\mu_A = \mu_Z$	$v_A = v_Z$	Case 1
x_{21}	$\mu_A = \mu_Z$	$v_A > v_Z$	Case 2.1
x_{22}	$\mu_A = \mu_Z$	$v_A < v_Z$	Case 2.2
x_{31}	$\mu_A > \mu_Z$	$v_A = v_Z$	Case 3.1
x_{32}	$\mu_A < \mu_Z$	$v_A = v_Z$	Case 3.2
x_{41}	$\mu_A > \mu_Z$	$v_A > v_Z$	Case 4.1
x_{42}	$\mu_A > \mu_Z$	$v_A < v_Z$	Case 4.2
x_{51}	$\mu_A < \mu_Z$	$v_A > v_Z$	Case 5.1
x_{52}	$\mu_A < \mu_Z$	$v_A < v_Z$	Case 5.2

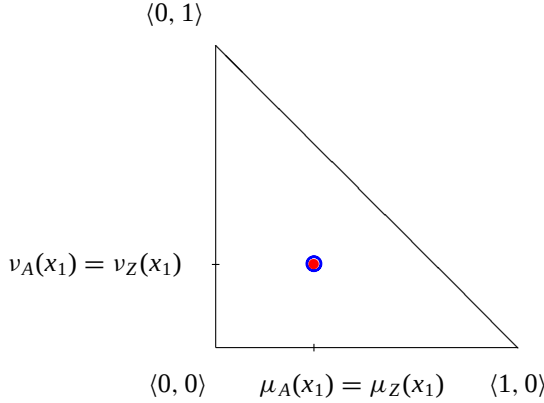


FIGURE 2

Case 1. The operator $X(A(x_1))$ coincides with the identity operator

Case 1. $\mu_A(x) = \mu_Z(x)$ and $\nu_A(x) = \nu_Z(x)$. Let us denote the set of all such $x \in E$ by E_1 . Then, for each $x_1 \in E_1$, we can set $\mu_B(x_1) = \nu_F(x_1) = 1$, $\nu_B(x_1) = \mu_C(x_1) = \nu_C(x_1) = \mu_F(x_1) = 0$, i.e., it is fulfilled that (see Fig. 2)

$$\begin{aligned}
 & X_{\{\langle x_1, 1, 0 \rangle\}, \{\langle x_1, 0, 0 \rangle\}, D(x_1), \{\langle x_1, 0, 1 \rangle\}, G(x_1)}(A(x_1)) \\
 &= \langle x_1, 1 \times \mu_A(x_1) + 0 \times (1 - \mu_A(x_1) - \mu_D(x_1)\nu_A(x_1)), \\
 &\quad 1 \times \nu_A(x_1) + 0 \times (1 - \nu_A(x_1) - \nu_G(x_1)\mu_A(x_1)) \rangle \\
 &= \langle x, \mu_Z(x_1), \nu_Z(x_1) \rangle \\
 &= Z(x_1),
 \end{aligned}$$

where $D(x_1)$, $G(x_1)$ may be arbitrary IFPs related to x_1 .

We see that (1) and (2) are fulfilled, because

$$\begin{aligned}
 0 &\leq \mu_B(x_1) + \nu_C(x_1) - \nu_C(x_1)\nu_G(x_1) = 1 + 0 - 0 \times \nu_G(x_1) \leq 1, \\
 0 &\leq \mu_C(x_1) + \nu_F(x_1) - \mu_C(x_1)\mu_D(x_1) = 0 + 1 - 0 \times \mu_D(x_1) \leq 1.
 \end{aligned}$$

Case 2.1. $\mu_A(x) = \mu_Z(x)$ and $\nu_A(x) > \nu_Z(x)$. Let us denote the set of all such $x \in E$ by E_{21} . Then, for each $x_{21} \in E_{21}$, we can set $\mu_B(x_{21}) = 1$, $\nu_B(x_{21}) = \mu_C(x_{21}) = \nu_C(x_{21}) = 0$, $\nu_F(x_{21}) = \frac{\nu_Z(x_{21})}{\nu_A(x_{21})}$, i.e., it is fulfilled that

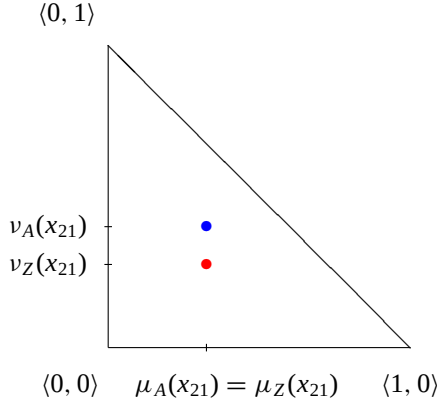


FIGURE 3

Case 2.1. The operator $X(A(x_{21}))$ moves the point $A(x)$ marked in blue vertically downwards

(see Fig. 3)

$$\begin{aligned}
 & X_{\{(x_{21}, 1, 0)\}, \{(x_{21}, 0, 0)\}, D(x_{21}), \left\{ \left\{ x_{21}, f(x_{21}), \frac{v_Z(x_{21})}{v_A(x_{21})} \right\} \right\}, G(x_{21})}(A(x_{21})) \\
 &= \langle x_{21}, 1 \times \mu_A(x_{21}) + 0 \times (1 - \mu_A(x_{21}) - \mu_D(x_{21})v_A(x_{21})), \\
 &\quad \frac{v_Z(x_{21})}{v_A(x_{21})}v_A(x_{21}) + 0 \times (1 - v_A(x_{21}) - v_G(x_{21})\mu_A(x_{21})) \rangle \\
 &= \langle x_{21}, \mu_Z(x_{21}), v_Z(x_{21}) \rangle \\
 &= Z(x_{21}),
 \end{aligned}$$

where $D(x_{21})$, $G(x_{21})$ may be arbitrary IFPs related to x_{21} and $0 \leq f(x_{21}) \leq 1 - \frac{v_Z(x)}{v_A(x)}$.

We see that (1) and (2) are fulfilled, because

$$\begin{aligned}
 0 &\leq \mu_B(x_{21}) + v_C(x_{21}) - v_C(x_{21})v_G(x_{21}) = 1 + 0 - 0 \times v_G(x_{21}) \leq 1, \\
 0 &\leq \mu_C(x_{21}) + v_F(x_{21}) - \mu_C(x_{21})\mu_D(x_{21}) = 0 + 1 - 0 \times \mu_D(x_{21}) \leq 1.
 \end{aligned}$$

Case 2.2. $\mu_A(x) = \mu_Z(x)$ and $v_A(x) < v_Z(x)$. Let us denote the set of all such $x \in E$ by E_{22} . Then, for each $x_{22} \in E_{22}$, we can set $\mu_B(x_{22}) = \mu_C(x_{22}) = \mu_Z(x_{22})$, $v_B(x_{22}) = \mu_D(x_{22}) = v_G(x_{22}) = 0$,

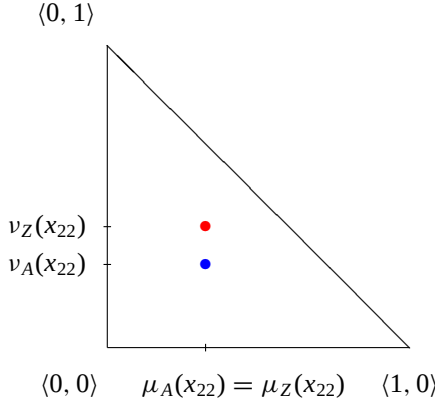


FIGURE 4

Case 2.2. The operator $X(A(x_{22}))$ moves the point $A(x)$ marked in blue vertically upwards

$\nu_C(x_{22}) = \nu_F(x_{22}) = \nu_Z(x_{22})$, i.e., it is fulfilled that (see Fig. 4)

$$\begin{aligned}
 & X_{\{(\langle x_{22}, \mu_Z(x_{22}), 0 \rangle), \{(\langle x_{22}, \mu_Z(x_{22}), \nu_Z(x_{22}) \rangle), \{(\langle x_{22}, 0, d(x_{22}) \rangle), \{(\langle x_{22}, f(x_{22}), \nu_Z(x_{22}) \rangle), \{(\langle x_{22}, g(x_{22}), 0 \rangle)\}}(A(x_{22})) \\
 &= \langle x_{22}, \mu_Z(x_{22}) \times \mu_A(x_{22}) + \mu_Z(x_{22}) \times (1 - \mu_A(x_{22}) - 0 \times \nu_A(x_{22})), \\
 &\quad \nu_Z(x_{22})\nu_A(x_{22}) + \nu_Z(x_{22}) \times (1 - \nu_A(x_{22}) - 0 \times \mu_A(x_{22})) \rangle \\
 &= \langle x_{22}, \mu_Z(x_{22}), \nu_Z(x_{22}) \rangle \\
 &= Z(x_{22}),
 \end{aligned}$$

where $0 \leq d(x_{22}) \leq 1$, $0 \leq f(x_{22}) \leq 1 - \nu_Z(x_{22})$, and $0 \leq g(x_{22}) \leq 1$.

We see that (1) and (2) are fulfilled, because

$$\begin{aligned}
 0 &\leq \mu_B(x_{22}) + \nu_C(x_{22}) - \nu_C(x_{22})\nu_G(x_{22}) = \mu_Z(x_{22}) + \nu_Z(x_{22}) - 0 \times \nu_G(x_{22}) \leq 1, \\
 0 &\leq \mu_C(x_{22}) + \nu_F(x_{22}) - \mu_C(x_{22})\mu_D(x_{22}) = \mu_Z(x_{22}) + \nu_Z(x_{22}) - \mu_Z(x_{22}) \times 0 \leq 1.
 \end{aligned}$$

Case 3.1. $\mu_A(x) > \mu_Z(x)$ and $\nu_A(x) = \nu_Z(x)$. Let us denote the set of all such $x \in E$ by E_{31} . Then, for each $x_{31} \in E_{31}$, we can set $\mu_B(x_{31}) = \frac{\mu_Z(x_{31})}{\mu_A(x_{31})}$, $\mu_C(x_{31}) = \nu_C(x_{31}) = \mu_F(x_{31}) = 0$, $\nu_F(x_{31}) = 1$, i.e., it is fulfilled that

$$\begin{aligned}
 & X_{\{(\langle x_{31}, \frac{\mu_Z(x_{31})}{\mu_A(x_{31})}, b(x_{31}) \rangle), \{(\langle x_{31}, 0, 0 \rangle), D(x_{31}), \{(\langle x_{31}, 0, 1 \rangle), G(x_{31})\}}(A(x_{31})) \\
 &= \langle x_{31}, \frac{\mu_Z(x_{31})}{\mu_A(x_{31})} \times \mu_A(x_{31}) + 0 \times (1 - \mu_A(x_{31}) - \mu_D(x_{31}) \times \nu_A(x_{31})), \\
 &\quad 1 \times \nu_Z(x_{31}) + 0 \times (1 - \nu_A(x_{31}) - \nu_G(x_{31}) \times \mu_A(x_{31})) \rangle \\
 &= \langle x_{31}, \mu_Z(x_{31}), \nu_Z(x_{31}) \rangle \\
 &= Z(x_{31}),
 \end{aligned}$$

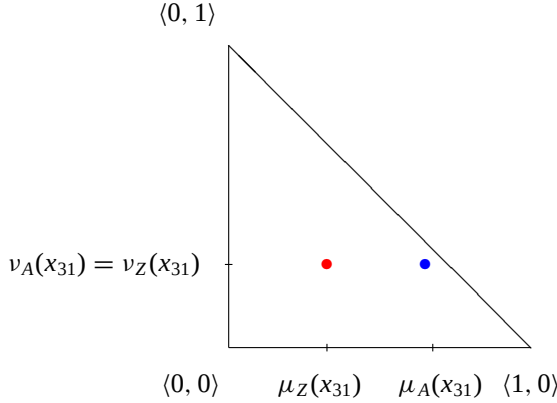


FIGURE 5

Case 3.1. The operator $X(A(x_{31}))$ moves the point $A(x)$ marked in blue horizontally to the left

where $0 \leq b(x_{31}) \leq 1 - \frac{\mu_Z(x_{31})}{\mu_A(x_{31})}$, and $D(x_{31})$, $G(x_{31})$ may be arbitrary IFPs related to x_{31} .

We see that (1) and (2) are fulfilled, because

$$0 \leq \mu_B(x_{31}) + \nu_C(x_{31}) - \nu_C(x_{31})\nu_G(x_{31}) = \frac{\mu_Z(x_{31})}{\mu_A(x_{31})} + 0 - 0 \times \nu_G(x_{31}) \leq 1,$$

$$0 \leq \mu_C(x_{31}) + \nu_F(x_{31}) - \mu_C(x_{31})\mu_D(x_{31}) = 0 + 1 - 0 \times \mu_D(x_{31}) \leq 1.$$

Case 3.2. $\mu_A(x) < \mu_Z(x)$ and $\nu_A(x) = \nu_Z(x)$. Let us denote the set of all such $x \in E$ by E_{32} . Then, for each $x_{32} \in E_{32}$, we can set $\mu_B(x_{32}) = \mu_C(x_{32}) = \mu_Z(x_{32})$, $\nu_C(x_{32}) = \nu_F(x_{32}) = \nu_Z(x_{32})$, $\mu_D(x_{32}) = \nu_G(x_{32}) = 0$, i.e., it is fulfilled that (see Fig. 6)

$$\begin{aligned} & X_{\{ \langle x_{32}, \mu_Z(x_{32}), b(x_{32}) \rangle \}, \{ \langle x_{32}, \mu_Z(x_{32}), \nu_Z(x_{32}) \rangle \}, \{ \langle x_{32}, 0, d(x_{32}) \rangle \}, \{ \langle x_{32}, f(x_{32}), \nu_Z(x_{32}) \rangle \}, \{ \langle x_{32}, g(x_{32}), 0 \rangle \}} (A(x_{32})) \\ &= \langle x_{32}, \mu_Z(x_{32}) \times \mu_A(x_{32}) + \mu_Z(x_{32}) \times (1 - \mu_A(x_{32}) - 0 \times \nu_A(x_{32})), \\ &\quad \nu_Z(x_{32}) \times \nu_A(x_{32}) + \nu_Z(x_{32}) \times (1 - \nu_A(x_{32}) - 0 \times \mu_A(x_{32})) \rangle \\ &= \langle x_{32}, \mu_Z(x_{32}), \nu_Z(x_{32}) \rangle \\ &= Z(x_{32}), \end{aligned}$$

where $0 \leq b(x_{32}) \leq 1 - \mu_Z(x_{32})$, $0 \leq d(x_{32}) \leq 1$, $0 \leq f(x_{32}) \leq 1 - \nu_Z(x_{32})$, $0 \leq g(x_{32}) \leq 1$.

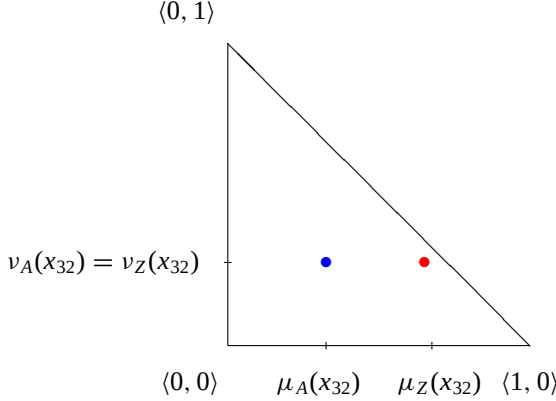


FIGURE 6

Case 3.2. The operator $X(A(x_{32}))$ moves the point $A(x)$ marked in blue horizontally to the right

We see that (1) and (2) are fulfilled, because

$$0 \leq \mu_B(x_{32}) + v_C(x_{32}) - v_C(x_{32})v_G(x_{32}) = \mu_Z(x_{32}) + v_Z(x_{32}) - v_Z(x_{32}) \times 0 \leq 1,$$

$$0 \leq \mu_C(x_{32}) + v_F(x_{32}) - \mu_C(x_{32})\mu_D(x_{32}) = \mu_Z(x_{32}) + v_Z(x_{32}) - \mu_Z(x_{32}) \times 0 \leq 1.$$

Case 4.1. $\mu_A(x) > \mu_Z(x)$ and $v_A(x) > v_Z(x)$. Let us denote the set of all such $x \in E$ by E_{41} . Then, for each $x_{41} \in E_{41}$, we can set $\mu_B(x_{41}) = \frac{\mu_Z(x_{41})}{\mu_A(x_{41})}$, $\mu_C(x_{41}) = v_C(x_{41}) = 0$, $v_F(x_{41}) = \frac{v_Z(x_{41})}{v_A(x_{41})}$, i.e., it is fulfilled that (see Fig. 7)

$$\begin{aligned} & X_{\{ \langle x_{41}, \frac{\mu_Z(x_{41})}{\mu_A(x_{41})}, b(x_{41}) \rangle, \{ \langle x_{41}, 0, 0 \rangle \}, D(x_{41}), \{ \langle x_{41}, f(x_{41}), \frac{v_Z(x_{41})}{v_A(x_{41})} \rangle \} \}} \cdot G(x_{41}) (A(x_{41})) \\ &= \langle x_{41}, \frac{\mu_Z(x_{41})}{\mu_A(x_{41})} \times \mu_A(x_{41}) + 0 \times (1 - \mu_A(x_{41}) - \mu_D(x_{41}) \times v_A(x_{41})), \\ &\quad \frac{v_Z(x_{41})}{v_A(x_{41})} \times v_A(x_{41}) + 0 \times (1 - v_A(x_{41}) - v_G(x_{41}) \times \mu_A(x_{41})) \rangle \\ &= \langle x_{41}, \mu_Z(x_{41}), v_Z(x_{41}) \rangle \\ &= Z(x_{41}), \end{aligned}$$

where $0 \leq b(x_{41}) \leq 1 - \frac{\mu_Z(x_{41})}{\mu_A(x_{41})}$, and $0 \leq f(x_{41}) \leq 1 - \frac{v_Z(x_{41})}{v_A(x_{41})}$, $D(x_{41})$, $G(x_{41})$ may be arbitrary IFPs related to x_{41} .

We see that (1) and (2) are fulfilled, because

$$0 \leq \mu_B(x_{41}) + v_C(x_{41}) - v_C(x_{41})v_G(x_{41}) = \frac{\mu_Z(x_{41})}{\mu_A(x_{41})} + 0 - v_G(x_{41}) \times 0 \leq 1,$$

$$0 \leq \mu_C(x_{41}) + v_F(x_{41}) - \mu_C(x_{41})\mu_D(x_{41}) = \frac{v_Z(x_{41})}{v_A(x_{41})} + 0 - \mu_D(x_{41}) \times 0 \leq 1.$$

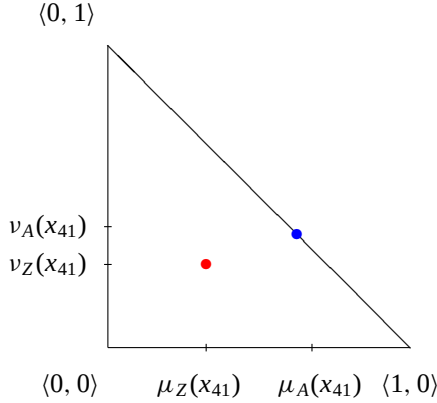


FIGURE 7

Case 4.1. The operator $X(A(x_{41}))$ moves the point $A(x)$ marked in blue downwards and to the left

Case 4.2. $\mu_A(x) > \mu_Z(x)$ and $v_A(x) < v_Z(x)$. Let us denote the set of all such $x \in E$ by E_{42} . Then, for each $x_{42} \in E_{42}$, we can set $\mu_B(x_{42}) = \mu_C(x_{42}) = \mu_Z(x_{42})$, $\mu_D(x_{42}) = v_G(x_{42}) = 0$, $v_C(x_{42}) = v_F(x_{42}) = v_Z(x_{42})$, i.e., it is fulfilled that (see Fig. 8)

$$\begin{aligned}
 & X_{\{ \langle x_{42}, \mu_Z(x_{42}), b(x_{42}) \rangle \}, \{ \langle x_{42}, \mu_Z(x_{42}), v_Z(x_{42}) \rangle \}, \{ \langle x_{42}, 0, d(x_{42}) \rangle \}, \{ \langle x_{42}, f(x_{42}), v_Z(x_{42}) \rangle \}, \{ \langle x_{42}, g(x_{42}), 0 \rangle \}} (A(x_{42})) \\
 &= \langle x_{42}, \mu_Z(x_{42}) \times \mu_A(x_{42}) + \mu_Z(x_{42}) \times (1 - \mu_A(x_{42}) - 0 \times v_A(x_{42})), \\
 &\quad v_Z(x_{42}) \times v_A(x_{42}) + v_Z(x_{42}) \times (1 - v_A(x_{42}) - 0 \times \mu_A(x_{42})) \rangle \\
 &= \langle x_{42}, \mu_Z(x_{42}), v_Z(x_{42}) \rangle \\
 &= Z(x_{42}),
 \end{aligned}$$

where $0 \leq b(x_{42}) \leq 1 - \mu_Z(x_{42})$, $0 \leq d(x_{42}) \leq 1$, $0 \leq f(x_{42}) \leq 1 - v_Z(x_{42})$, $0 \leq g(x_{42}) \leq 1$.

We see that (1) and (2) are fulfilled, because

$$\begin{aligned}
 0 &\leq \mu_B(x_{42}) + v_C(x_{42}) - v_C(x_{42})v_G(x_{42}) = \mu_Z(x_{42}) + v_Z(x_{42}) - 0 \times v_Z(x_{42}) \leq 1, \\
 0 &\leq \mu_C(x_{42}) + v_F(x_{42}) - \mu_C(x_{42})\mu_D(x_{42}) = \mu_Z(x_{42}) + v_Z(x_{42}) - \mu_Z(x_{42}) \times 0 \leq 1.
 \end{aligned}$$

Case 5.1. $\mu_A(x) < \mu_Z(x)$ and $v_A(x) > v_Z(x)$. Let us denote the set of all such $x \in E$ by E_{51} . Then, for each $x_{51} \in E_{51}$, we can set $\mu_B(x_{51}) = \mu_C(x_{51}) = \mu_Z(x_{51})$, $\mu_D(x_{51}) = v_G(x_{51}) = 0$, $v_C(x_{51}) = v_F(x_{51}) = v_Z(x_{51})$,

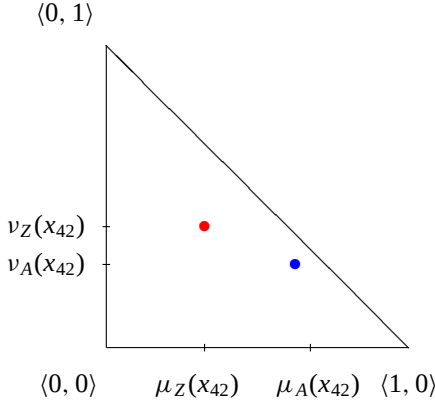


FIGURE 8

Case 4.2. The operator $X(A(x_{42}))$ moves the point $A(x)$ marked in blue upwards and to the left

i.e., it is fulfilled that (see Fig. 9)

$$\begin{aligned}
 & X_{\{(\langle x_{51}, \mu_Z(x_{51}), b(x_{51}) \rangle), \{(\langle x_{51}, \mu_Z(x_{51}), \nu_Z(x_{51}) \rangle), \{(\langle x_{51}, 0, d(x_{51}) \rangle), \{(\langle x_{51}, f(x_{51}), \nu_Z(x_{51}) \rangle), \{(\langle x_{51}, g(x_{51}), 0 \rangle)\}}(A(x_{51})) \\
 &= \langle x_{51}, \mu_Z(x_{51})\mu_A(x_{51}) + \mu_Z(x_{51})(1 - \mu_A(x_{51}) - \nu_A(x_{51}) \times 0), \\
 &\quad \nu_Z(x_{51})\nu_A(x_{51}) + \nu_Z(x_{51})(1 - \nu_A(x_{51}) - \mu_A(x_{51}) \times 0) \\
 &= \langle x_{51}, \mu_Z(x_{51}), \nu_Z(x_{51}) \rangle \\
 &= Z(x_{51}),
 \end{aligned}$$

where $0 \leq b(x_{51}) \leq 1 - \mu_Z(x_{51})$, $0 \leq d(x_{51}) \leq 1$, $0 \leq f(x_{51}) \leq 1 - \nu_Z(x_{51})$, and $0 \leq g(x_{51}) \leq 1$.

We see that (1) and (2) are fulfilled, because

$$0 \leq \mu_B(x_{51}) + \nu_C(x_{51}) - \nu_C(x_{51})\nu_G(x_{51}) = \mu_Z(x_{51}) + \nu_Z(x_{51}) - \nu_Z(x_{51}) \times 0 \leq 1,$$

$$0 \leq \mu_C(x_{51}) + \nu_F(x_{51}) - \mu_C(x_{51})\mu_D(x_{51}) = \mu_Z(x_{51}) + \nu_Z(x_{51}) - \nu_Z(x_{51}) \times 0 \leq 1.$$

Case 5.2. $\mu_A(x) < \mu_Z(x)$ and $\nu_A(x) < \nu_Z(x)$. Let us denote the set of all such $x \in E$ by E_{52} . Then, for each $x_{52} \in E_{52}$, we can set $\mu_B(x_{52}) = \mu_C(x_{52}) = \mu_Z(x_{52})$, $\mu_D(x_{52}) = \nu_G(x_{52}) = 0$, $\nu_C(x_{52}) = \nu_F(x_{52}) = \nu_Z(x_{52})$,

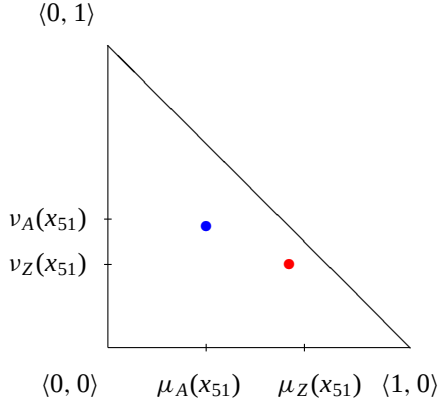


FIGURE 9

Case 5.1. The operator $X(A(x_{51}))$ moves the point $A(x)$ marked in blue downwards and to the right

i.e., it is fulfilled that (see Fig. 10)

$$\begin{aligned}
 & X_{\{\langle x_{52}, \mu_Z(x_{52}), b(x_{52}) \rangle, \langle x_{52}, \mu_Z(x_{52}), \nu_Z(x_{52}) \rangle, \langle x_{52}, 0, d(x_{52}) \rangle, \langle x_{52}, f(x_{52}), \nu_Z(x_{52}) \rangle, \langle x_{52}, g(x_{52}), 0 \rangle\}}(A(x_{52})) \\
 &= \langle x_{52}, \mu_Z(x_{52})\mu_A(x_{52}) + \mu_Z(x_{52})(1 - \mu_A(x_{52}) - \nu_A(x_{52}) \times 0), \\
 &\quad \nu_Z(x_{52})\nu_A(x_{52}) + \nu_Z(x_{52})(1 - \nu_A(x_{52}) - \mu_A(x_{52}) \times 0) \\
 &= \langle x_{52}, \mu_Z(x_{52}), \nu_Z(x_{52}) \rangle \\
 &= Z(x_{52}),
 \end{aligned}$$

where $0 \leq b(x_{52}) \leq 1 - \mu_Z(x_{52})$, $0 \leq d(x_{52}) \leq 1$, $0 \leq f(x_{52}) \leq 1 - \nu_Z(x_{52})$, and $0 \leq g(x_{52}) \leq 1$.

We immediately see that (1) and (2) are fulfilled.

$$0 \leq \mu_B(x_{52}) + \nu_C(x_{52}) - \nu_C(x_{52})\nu_G(x_{52}) = \mu_Z(x_{52}) + \nu_Z(x_{52}) - \nu_Z(x_{52}) \times 0 \leq 1,$$

$$0 \leq \mu_C(x_{52}) + \nu_F(x_{52}) - \mu_C(x_{52})\mu_D(x_{52}) = \mu_Z(x_{52}) + \nu_Z(x_{52}) - \nu_Z(x_{52}) \times 0 \leq 1.$$

Now the theorem follows from the fact that each $x \in E$ must belong exactly to one of the sets $E_1, E_{21}, E_{22}, E_{31}, E_{32}, E_{41}, E_{42}, E_{51}, E_{52}$, since

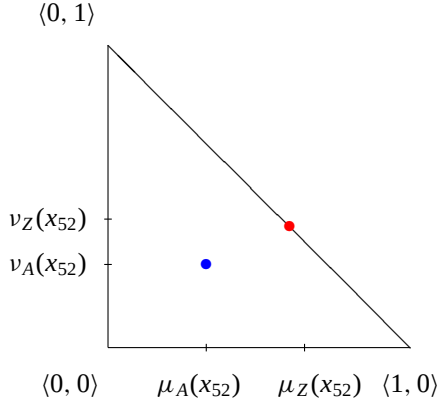


FIGURE 10

Case 5.2. The operator $X(A(x_{52}))$ moves the point $A(x)$ marked in blue upwards and to the right

we have:

$$E = E_1 \cup E_{21} \cup E_{22} \cup E_{31} \cup E_{32} \cup E_{41} \cup E_{42} \cup E_{51} \cup E_{52}.$$

and

$$E_i \cap E_j = \emptyset \text{ for } i \neq j.$$

Therefore, we can construct the IFSs B, C, D, F, G as follows:

$$B = \{\langle x, \mu_B(x), \nu_B(x) \rangle | x \in E\},$$

where

$$\mu_B(x) = \begin{cases} 1, & \text{if } x \in E_1 \cup E_{21} \\ \mu_Z(x), & \text{if } x \in E_{22} \cup E_{32} \cup E_{42} \cup E_{51} \cup E_{52} \\ \frac{\mu_Z(x)}{\mu_A(x)}, & \text{if } x \in E_{31} \cup E_{41} \end{cases}$$

$$\nu_B(x) = \begin{cases} 0, & \text{if } x \in E_1 \cup E_{21} \cup E_{22} \\ b, & \text{if } x \in E_{31} \cup E_{32} \cup E_{41} \cup E_{42} \cup E_{51} \cup E_{52}, \end{cases}$$

where b satisfies the above mentioned conditions;

$$C = \{\langle x, \mu_C(x), \nu_C(x) \rangle | x \in E\},$$

where

$$\begin{aligned} \mu_C(x) &= \begin{cases} 0, & \text{if } x \in E_1 \cup E_{21} \cup E_{31} \cup E_{41} \\ \mu_Z(x), & \text{if } x \in E_{22} \cup E_{32} \cup E_{42} \cup E_{51} \cup E_{52} \end{cases} \\ \nu_C(x) &= \begin{cases} 0, & \text{if } x \in E_1 \cup E_{21} \cup E_{31} \cup E_{41} \\ \nu_Z(x), & \text{if } x \in E_{22} \cup E_{32} \cup E_{42} \cup E_{51} \cup E_{52}; \end{cases} \\ D &= \{\langle x, \mu_D(x), \nu_D(x) \rangle | x \in E\}, \end{aligned}$$

where

$$\begin{aligned} \mu_D(x) &= \begin{cases} d_1, & \text{if } x \in E_1 \cup E_{21} \cup E_{31} \cup E_{41} \\ 0, & \text{if } x \in E_{22} \cup E_{32} \cup E_{42} \cup E_{51} \cup E_{52} \end{cases} \\ \nu_D(x) &= \begin{cases} d_2, & \text{if } x \in E_1 \cup E_{21} \cup E_{31} \cup E_{41} \\ d_3, & \text{if } x \in E_{22} \cup E_{32} \cup E_{42} \cup E_{51} \cup E_{52} \end{cases} \end{aligned}$$

where $d_1, d_2, d_3 \in [0, 1]$, $d_1 + d_2 \leq 1$ are arbitrary real numbers and d_3 satisfies the above mentioned conditions;

$$F = \{\langle x, \mu_F(x), \nu_F(x) \rangle | x \in E\},$$

where

$$\begin{aligned} \mu_F(x) &= \begin{cases} 0, & \text{if } x \in E_1 \cup E_{31} \\ f, & \text{if } x \in E_{21} \cup E_{31} \cup E_{32} \cup E_{41} \cup E_{42} \cup E_{51} \cup E_{52}, \end{cases} \\ \nu_F(x) &= \begin{cases} 1, & \text{if } x \in E_1 \cup E_{31} \\ \frac{\nu_Z(x)}{\nu_A(x)}, & \text{if } x \in E_{21} \cup E_{41} \\ \nu_Z(x), & \text{if } x \in E_{22} \cup E_{32} \cup E_{42} \cup E_{51} \cup E_{52} \end{cases} \end{aligned}$$

where f satisfies the above mentioned conditions;

$$G = \{\langle x, \mu_G(x), \nu_G(x) \rangle | x \in E\},$$

where

$$\mu_G(x) = \begin{cases} g_1, & \text{if } x \in E_1 \cup E_{21} \cup E_{31} \cup E_{41} \\ g_2, & \text{if } x \in E_{22} \cup E_{32} \cup E_{42} \cup E_{51} \cup E_{52} \end{cases}$$

$$\nu_G(x) = \begin{cases} g_3, & \text{if } x \in E_1 \cup E_{21} \cup E_{31} \cup E_{41} \\ 0, & \text{if } x \in E_{22} \cup E_{32} \cup E_{42} \cup E_{51} \cup E_{52} \end{cases}$$

where $g_1, g_2, g_3 \in [0, 1]$, $g_1 + g_3 \leq 1$ are arbitrary real numbers and g_2 satisfies the above mentioned conditions;

Hence, for each element $x \in E$ we found the respective values for the degrees of membership and non-membership of the IFSs B, C, D, F, G so that (3) is valid.

Because each one of the sets B, C, D, F, G is defined over the set E , we constructed membership and non-membership functions for each one of these sets, i.e., we constructed IFSs that satisfy (3), with which the theorem is proved.

Remark 1. Evidently the proposed construction provides only some among the many possible sets that satisfy (3). To illustrate this point further let us consider the IFSs

$$\langle x, \mu_A(x), \nu_A(x) \rangle = \left\langle x, \frac{1}{4}, \frac{17}{40} \right\rangle, \quad \langle x, \mu_Z(x), \nu_Z(x) \rangle = \left\langle x, \frac{1}{2}, \frac{1}{4} \right\rangle.$$

From Case 5.2 it follows, that the IFSs $B = \{\langle x, \frac{1}{2}, 0 \rangle\}$, $C = \{\langle x, \frac{1}{2}, \frac{1}{4} \rangle\}$, $D = \{\langle x, 0, 1 \rangle\}$, $F = \{\langle x, 0, \frac{1}{4} \rangle\}$, $G = \{\langle x, 1, 0 \rangle\}$ are such that $X_{B,C,D,F,G}(A) = Z$.

However, the same is also true for the following IFSs $\tilde{B} = \langle x, 1, 0 \rangle$, $\tilde{C} = \langle x, \frac{1}{3}, 0 \rangle$, $\tilde{D} = \tilde{G} = \langle x, 0, 0 \rangle$, $\tilde{F} = \langle x, 0, \frac{10}{17} \rangle$, and obviously $\tilde{B} \neq B$; $\tilde{C} \neq C$; $\tilde{D} \neq D$; $\tilde{G} \neq G$; $\tilde{F} \neq F$.

4 CONCLUSION

In the last few years various modifications of IFS have appeared claiming to be extensions of IFS or interval-valued IFS - see, for example, [25,32]. In [29]

there are proofs that these sets are representable by standard IFSs. The new operator will provide additional opportunities to prove that some of the sets claiming to be extensions of IFSs can be transformed by the new operator to a standard IFS. For example, in [18] it is proved that the vague sets [21] are IFS. Using the X -operator we can give another proof of this fact.

We finish with the following **Open problems**:

1. Which other sets can be arguments of the X -operator?
2. Can the operator $X_{B,C,D,F,G}$ be represented as composition of some of the remaining operators?

ACKNOWLEDGEMENTS

K.A. and J. K. would like to thank project IC-PL/14/2024-2025 for the kind support. P. V. and V. A. are grateful for the support provided by the Bulgarian National Science Fund under Grant Ref. No. KP-06-N72/8 “Intuitionistic fuzzy methods for data analysis with an emphasis on the blood donation system in Bulgaria”.

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